1 Countability: True or False

Note 11

(a) The set of all irrational numbers $\mathbb{R} \backslash \mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) f and g are injective (one-to-one) $\implies$ $f \circ g$ is injective (one-to-one).

(e) f is surjective (onto) $\implies$ $f \circ g$ is surjective (onto).

2 Proof Debugging

Note 11

Find the precise error in the following proof:

False Claim: The set of rationals r such that $0 \le r \le 1$ is uncountable.

Proof? Suppose towards a contradiction that there is a bijection $f : \mathbb{N} \to \mathbb{Q}[0,1]$, where $\mathbb{Q}[0,1]$ denotes the rationals in $[0,1]$. This allows us to list all the rationals between 0 and 1, with the j-th element of the list being $f(j)$. Suppose we represent each of these rationals by their non-terminating expansion (for example, $0.4999\ldots$ rather than 0.5).

Let d_j denote the j-th digit of $f(j)$. We define a new number e, whose j-th digit e_j is equal to $(d_j + 2) \pmod{10}$. We claim that e does not occur in our list of rationals between 0 and 1. This is because e cannot be equal to $f(j)$ for any j, since it differs from $f(j)$ in the j-th digit by more than 1.

This is a contradiction, so therefore the set of rationals between 0 and 1 is uncountable.

3 Countability Basics

Note 11

(a) Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$, an injection (one-to-one)? Briefly justify.

(b) Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$, a surjection (onto)? Briefly justify.

(c) The Bernstein-Schroder theorem states that, if there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then a bijection exists between A and B. Use this to demonstrate that $(0,1)$ and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality by defining appropriate injections.

4 Counting Cartesian Products

Note 11

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.