CS 70 Discrete Mathematics and Probability Theory
Spring 2023 Satish Rao and Babak Ayazifar **DIS 11A**

1 Beast Arcade

Note 15

One day you find yourself inside the Mr. Beast Arcade, which is full of games that pay YOU to play them!

(a) In the first game, Chandler hands you a crisp \$20 bill up front. Then, he flips a coin that shows heads with probability p repeatedly, stopping when a heads comes up for the first time. You receive an additional dollar for each flip. How much money will you get in expectation?

(b) In the next game, Karl rolls a fair 6-sided die. He then calculates 2^x, where x is the result of that die and hands you that much money. What is the expected amount of money you'll receive?

(c) For the last game, Jimmy makes your friend flip a fair coin $10,000$ times in a row, keeping track of the number of heads that show up. He then hands you a briefcase filled with $\$1,000$ and says he will also pay you \$5 for each head that comes up. Let X be a random variable representing the number of heads your friend flips. Use it to come up with an expression for Y, a random variable representing the total amount of money you'll receive.

(d) What is $\mathbb{E}[Y]$? What about $\mathbb{P}[Y = 26,000]$?

2 Variance

Note 16

(a) Let X be a random variable representing the outcome of the roll of one fair 6-sided die. What is $\text{Var}(X)$?

(b) Let Z be a random variable representing the average of n rolls of a fair 6-sided die. What is $\text{Var}(Z)$?

3 Diversify Your Hand

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

Note 15
Note 16

(a) Calculate $\mathbb{E}[X]$. (Hint: Consider indicator variables X_i representing whether i appears in the hand.)

(b) Calculate $\text{Var}(X)$.

4 Family Planning

Note 15

Mr. and Mrs. Johnson decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Johnsons have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C=1$	$C=2$	$C=3$
$G=0$			
$G=1$			

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}[G=0]$	
$\mathbb{P}[G=1]$	

$\mathbb{P}[C=1]$	$\mathbb{P}[C=2]$	$\mathbb{P}[C=3]$

(d) Are G and C independent?

(e) What is the expected number of girls the Johnsons will have? What is the expected number of children that the Johnsons will have?