Due: Saturday, 1/28, 4:00 PM
Grace period until Saturday, 1/28, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Calculus Review

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t\,.$$

(b) Compute the values of $x \in (-2,2)$ that correspond to local maxima and minima of the function

$$f(x) = \int_0^{x^2} t\cos\left(\sqrt{t}\right)\mathrm{d}t\,.$$

Classify which x correspond to local maxima and which to local minima.

(c) Compute the double integral

$$\iint_R 2x + y\,\mathrm{d}A\,,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

2 Prove or Disprove

Note 2 For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

3 Rationals and Irrationals

Note 2 Prove that the product of a non-zero rational number and an irrational number is irrational.

4 Twin Primes

Note 2 (a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

5 Airport

Note 3 Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

6 Proving Inequality

Note 3 For all positive integers $n \geq 1$, prove that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we would like you to use induction. If you're having trouble with the inductive step, try strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

7 AM-GM

Note 3 For nonnegative real numbers $a_1, \cdots, a_n$, the arithmetic mean, or average, is defined by

$$\frac{a_1 + \cdots + a_n}{n},$$

and the geometric mean is defined by

$$\sqrt[n]{a_1 \cdots a_n}.$$

In this problem, we will prove the "AM-GM" inequality. More precisely, for all positive integers $n \geq 2$, given any nonnegative real numbers $a_1, \cdots, a_n$, we will show that

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

We will do so by induction on n, but in an unusual way.

(a) Prove that the inequality holds for $n = 2$. In other words, for nonnegative real numbers a_1 and a_2, show that

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

(This equation might be of use: $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$)

(b) For some positive integer k, suppose that the AM-GM inequality holds for $n = 2^k$. Show that the AM-GM inequality holds for $n = 2^{k+1}$. (Hint: Think about how the AM-GM inequality for $n = 2$ could be used here.)

(c) For some positive integer $k \geq 2$, suppose that the AM-GM inequality holds for $n = k$. Show that the AM-GM inequality holds for $n = k - 1$. (Hint: In the AM-GM expression for $n = k$, consider substituting $a_k = \frac{a_1 + \cdots + a_{k-1}}{k-1}$.)

(d) Argue why parts (a) - (c) imply that the AM-GM inequality holds for all positive integers $n \geq 2$.

8 A Coin Game

Note 3

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)