CS 70 Spring 2023 | Discrete Mathematics and Probability Theory
Satish Rao and Babak Ayazifar | HW 03

Due: Saturday, 2/11, 4:00 PM
Grace period until Saturday, 2/11, 6:00 PM

## Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

## 1 Planarity and Graph Complements

Note 5

Let $G = (V,E)$ be an undirected graph. We define the complement of $G$ as $\overline{G} = (V,\overline{E})$ where $\overline{E} = \{(i,j) \mid i,j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as $G$, but an edge $e$ exists is $\overline{G}$ if and only if it does not exist in $G$.

(a) Suppose $G$ has $v$ vertices and $e$ edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 15 vertices, $G$ being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph $G$ with at least 15 vertices, if $\overline{G}$ is non-planar, then $G$ is planar. Construct a counterexample to show that the converse does not hold.

*Hint: Recall that if a graph contains a copy of $K_5$, then it is non-planar. Can this fact be used to construct a counterexample?*

## 2 Touring Hypercube

Note 5

In the lecture, you have seen that if $G$ is a hypercube of dimension $n$, then

- The vertices of $G$ are the binary strings of length $n$.
- $u$ and $v$ are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.

- Each pair of consecutive vertices is connected by an edge.
- $v_0$ and $v_k$ are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if $n$ is even.

(b) Show that every hypercube has a Hamiltonian tour.

## 3 Binary Trees

Note 5

You may have seen the recursive definition of binary trees from previous classes. Here, we define binary trees in graph theoretic terms as follows (**Note:** here we will modify the definition of leaves slightly for consistency).

- A binary tree of height $> 0$ is a tree where exactly one vertex, called the **root**, has degree 2, and all other vertices have degrees 1 or 3. Each vertex of degree 1 is called a **leaf**. The **height** $h$ is defined as the maximum length of the path between the root and any leaf.
- A binary tree of height 0 is the graph with a single vertex. The vertex is both a leaf and a root.

(a) Let $T$ be a binary tree of height $> 0$, and let $h(T)$ denote it's height. Let $r$ be the root in $T$ and $u$ and $v$ be it's neighbors. Show that removing $r$ from $T$ will result in two binary trees, $L, R$ with roots $u$ and $v$ respectively. Also, show that $h(T) = \max(h(L), h(R)) + 1$.

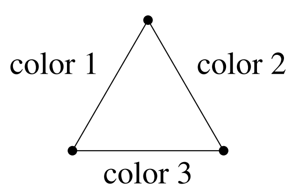
(b) Using the graph theoretic definition of binary trees, prove that the number of vertices in a binary tree of height $h$ is at most $2^{h+1} - 1$.

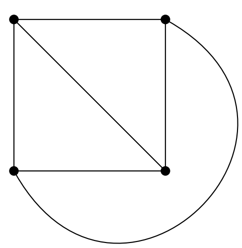
(c) Prove that all binary trees with $n$ leaves have $2n - 1$ vertices.

## 4 Edge Colorings

Note 5

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

(c) Show that a tree can be edge colored with $d$ colors where $d$ is the maximum degree of any vertex.

## 5 Modular Practice

Note 6

Solve the following modular arithmetic equations for $x$ and $y$.

(a) $9x+5 \equiv 7 \pmod{13}$.

(b) Show that $3x+12 \equiv 4 \pmod{21}$ does not have a solution.

(c) The system of simultaneous equations $5x+4y \equiv 0 \pmod{7}$ and $2x+y \equiv 4 \pmod{7}$.

(d) $13^{2023} \equiv x \pmod{12}$.

(e) $7^{62} \equiv x \pmod{11}$.

## 6 Nontrivial Modular Solutions

Note 6

(a) What are all the possible perfect cubes modulo 7?

(b) Show that any solution to $x^3+2y^3 \equiv 0 \pmod{7}$ must satisfy $x \equiv y \equiv 0 \pmod{7}$.

(c) Using part (b), prove that $x^3+2y^3 = 7x^2y$ has no non-trivial solutions $(x,y)$ in the integers. In other words, there are no integers $x$ and $y$, that satisfy this equation, except the trivial solution $x = y = 0$.

[*Hint:* Consider some nontrivial solution $(x,y)$ with the smallest value for $|x|$ (why are we allowed to consider this?). Then arrive at a contradiction by finding another solution $(x',y')$ with $|x'| < |x|$.]

## 7 Wilson's Theorem

Note 6

Wilson's Theorem states the following is true if and only if $p$ is prime:

$$(p-1)! \equiv -1 \pmod{p}.$$

Prove both directions (it holds if AND only if $p$ is prime).

Hint for the if direction: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$ to pair up terms with their inverses, when possible. What terms are left unpaired?

Hint for the only if direction: If $p$ is composite, then it has some prime factor $q$. What can we say about $(p-1)! \pmod{q}$?