CS 70 Spring 2023 — Discrete Mathematics and Probability Theory — HW 6

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Due: Saturday 3/4, 4:00 PM
Grace period until Saturday 3/4, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Unions and Intersections

Note 11

Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

2 Finite and Infinite Graphs

Note 5
Note 11

The graph material that we learned in lecture still applies if the set of vertices of a graph is infinite. We thus make a distinction between finite and infinite graphs: a graph $G = (V, E)$ is finite if V and E are both finite. Otherwise, the graph is infinite. As examples, consider the infinite graphs

- $G_1 = (V = \mathbb{N},\ E = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid |i - j| = 1\})$
- $G_2 = (V = \mathbb{Z},\ E = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid |i - j| = 1\})$
- $G_3 = (V = \mathbb{Z},\ E = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i < j\})$
- $G_4 = (V = \mathbb{Z}^2,\ E = \{((i, j),(k,l)) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \mid (i = k \wedge |j - l| = 1) \vee (j = l \wedge |i - k| = 1)\})$

Observe that G_1 is a line of natural numbers, G_2 is a line of integers, G_3 is a complete graph over all integers, and G_4 is a grid of integers.

Prove whether the following sets of graphs are countable or uncountable.

(a) The set of all finite graphs $G = (V,E)$, for $V \subseteq \mathbb{N}$

(b) The set of all graphs over a fixed, countably infinite set of vertices, where the degree of each vertex is exactly two. For instance, every vertex in G_2 (defined above) has degree 2.

(c) We say that graphs $G = (V,E)$ and $G' = (V',E')$ are isomorphic if the exists some bijection $f : V \to V'$ such that $(u, v) \in E$ iff $(f(u), f(v)) \in E'$. Such a bijection f is called a *graph isomorphism*. Suppose we consider two graphs to be the equivalent if they are isomorphic. The idea is that if we relabel the vertices of a graph, it is still the same graph.

Using this definition of "being the same graph", prove that the set of trees over countably infinite vertices is uncountable.

(*Hint:* Construct an injection from the set of infinite bitstrings to the set of infinite trees; G_1 could be a good starting point. It may help to show that for any graph isomorphism f, and any vertex v, the vertices $f(v)$ and v have the same degree.)

3 Countability Proof Practice

Note 11

(a) A disk is a 2D region of the form $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap. Is this set always countable, or potentially uncountable?
(*Hint*: Attempt to relate it to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$.)

(b) A circle is a subset of the plane of the form $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap. Is this set always countable, or potentially uncountable?
(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)

(c) Is the set containing all increasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \geq f(y)$) countable or uncountable? Prove your answer.

(d) Is the set containing all decreasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \leq f(y)$) countable or uncountable? Prove your answer.

4 Fixed Points

Note 12 Consider the problem of determining if a program P has any fixed points. Given any program P, a fixed point is an input x such that $P(x)$ outputs x.

(a) Prove that the problem of determining whether a program has a fixed point is uncomputable.

(b) Consider the problem of outputting a fixed point of a program if it has one, and outputting "Null" otherwise. Prove that this problem is uncomputable.

(c) Consider the problem of outputting a fixed point of a program F if the fixed point exists *and* is a natural number, and outputting “Null” otherwise.

Show that if this problem can be solved, then the problem in part (b) can be solved. What does this say about the computability of this problem? (You may assume that the set of all possible inputs to a program is countable, as is the case on your computer.)

5 Kolmogorov Complexity

Note 12 Compressing a bit string x of length n can be interpreted as the task of creating a program of fewer than n bits that returns x. The Kolmogorov complexity of a string $K(x)$ is the length of an optimally-compressed copy of x; that is, $K(x)$ is the length of shortest program that returns x.

(a) Explain why the notion of the "smallest positive integer that cannot be defined in under 280 characters" is paradoxical.

(b) Prove that for any length n, there is at least one string of bits that cannot be compressed to less than n bits.

(c) Say you have a program K that outputs the Kolmogorov complexity of any input string. Under the assumption that you can use such a program K as a subroutine, design another program P that takes an integer n as input, and outputs the length-n binary string with the highest Kolmogorov complexity. If there is more than one string with the highest complexity, output the one that comes first lexicographically.

(d) Let’s say you compile the program P you just wrote and get an m bit executable, for some $m \in \mathbb{N}$ (i.e. the program P can be represented in m bits). Prove that the program P (and consequently the program K) cannot exist.

(*Hint*: Consider what happens when P is given a very large input n.)

6 Counting, Counting, and More Counting

Note 10 The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 19-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

   i. How many different 13-card bridge hands are there?
   ii. How many different 13-card bridge hands are there that contain no aces?
   iii. How many different 13-card bridge hands are there that contain all four aces?
   iv. How many different 13-card bridge hands are there that contain exactly 4 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

   i. How many different anagrams of ALABAMA are there?
   ii. How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

   i. C is the left neighbor of E
   ii. C is on the left of E (and not necessarily E's neighbor)

(h) We have 8 balls, numbered 1 through 8, and 25 bins. How many different ways are there to distribute these 8 balls among the 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(i) How many different ways are there to throw 8 identical balls into 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(j) We throw 8 identical balls into 6 bins. How many different ways are there to distribute these 8 balls among the 6 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 6).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

7 Fermat's Wristband

Note 7
Note 10

Let p be a prime number and let n be a positive integer. We have beads of n different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there to construct such a sequence of p beads with up to n different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $n = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of n and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.