CS 70 Spring 2023 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — HW 7

Due: —
Grace period until —

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Shipping Crates

Note 10

A widget factory has four loading docks for storing crates of ready-to-ship widgets. Suppose the factory produces 8 indistinguishable crates of widgets and sends each crate to one of the four loading docks.

(a) How many ways are there to distribute the crates among the loading docks?

(b) Now, assume that any time a loading dock contains at least 5 crates, a truck picks up 5 crates from that dock and ships them away. (e.g., if 6 crates are sent to a loading dock, the truck removes 5, leaving 1 leftover crate still in the dock). We will now consider two configurations to be identical if, for every loading dock, the two configurations have the same number of leftover crates in that dock. How would your answer in the previous part compare to the number of outcomes given the new setup? Do not compute the actual value in this part, we will doing that in parts (c) - (e). We are looking for a qualitative answer (greater than part (a), equal to part (a), less than or equal to part (a), etc.) Justify your answer.

(c) We will now attempt to count the number of configurations of crates. First, we look at the case where crates are removed from the dock. How many ways are there to distribute the crates such that some crate gets removed from the dock?

(d) How many ways are there to distribute the crates such that no crates are removed from the dock; i.e. no dock receives at least 5 crates?

(e) Putting it together now, what are the total number of possible configurations for crates in the modified scenario? *Hint:* Observe that, regardless of which dock receives the 5 crates, we end up in the same situation. After all the shipping has been done, how many possible configurations of leftover crates in loading docks are there?

2 Grids and Trees!

Note 10 Suppose we are given an $n \times n$ grid, for $n \geq 1$, where one starts at $(0,0)$ and goes to (n,n). On this grid, we are only allowed to move left, right, up, or down by increments of 1.

(a) How many shortest paths are there that go from $(0,0)$ to (n,n)?

(b) How many shortest paths are there that go from $(0,0)$ to $(n-1,n+1)$?

Now, consider shortest paths that meet the conditions where we can only visit points (x,y) where $y \leq x$. That is, the path cannot cross line $y = x$. We call these paths n-legal paths for a maze of side length n. Let F_n be the number of n-legal paths.

(c) Compute the number of shortest paths from $(0,0)$ to (n,n) that cross $y = x$. (Hint: Let (i,i) be the first time the shortest path crosses the line $y = x$. Then the remaining path starts from $(i,i+1)$ and continues to (n,n). If in the remainder of the path one exchanges y-direction moves with x-direction moves and vice versa, where does one end up?)

(d) Compute the number of shortest paths from $(0,0)$ to (n,n) that do not cross $y = x$. (You may find your answers from parts (a) and (c) useful.)

(e) A different idea is to derive a recursive formula for the number of paths. Fix some i with $0 \leq i \leq n-1$. We wish to count the number of n-legal paths where the last time the path touches the line $y = x$ is the point (i,i). Show that the number of such paths is $F_i \cdot F_{n-i-1}$. (Hint: If $i = 0$, what are your first and last moves, and where is the remainder of the path allowed to go?)

(f) Explain why $F_n = \sum_{i=0}^{n-1} F_i \cdot F_{n-i-1}$.

(g) Create and explain a recursive formula for the number of trees with n vertices $(n \geq 1)$, where each non-root node has degree at most 3, and the root node has degree at most 2. Two trees are different if and only if either left-subtree is different or right-subtree is different.

(Notice something about your formula and the grid problem. Neat!)

3 Counting on Graphs + Symmetry

Note 10 (a) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(d) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

4 Proofs of the Combinatorial Variety

Note 10

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a, b, c) such that $a + b + c = m$.)

5 Fibonacci Fashion

Note 10

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1, 2, \ldots, n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1, 2, \ldots, n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \ge 0} \sum_{x_2 \ge 0} \cdots \sum_{x_t \ge 0} \binom{n}{x_1} \binom{n - x_1}{x_2} \binom{n - x_2}{x_3} \cdots \binom{n - x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)