Due: Saturday, 3/18, 4:00 PM
Grace period until Saturday, 3/18, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Probability Warm-Up

Note 13

(a) Suppose that we have a bucket of 30 green balls and 70 orange balls. If we pick 15 balls uniformly out of the bucket, what is the probability of getting exactly k green balls (assuming $0 \leq k \leq 15$) if the sampling is done **with** replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?

(b) Same as part (a), but the sampling is **without** replacement, i.e. after we take a ball out the bucket we **do not** return the ball back to the bucket.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

2 Five Up

Note 13

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

(a) What is the size of the sample space, $|\Omega|$?

(b) How many elements of Ω have exactly three heads?

(c) How many elements of Ω have three or more heads?

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

(d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?

(e) What is the chance of observing at least one head?

(f) What about the chance of observing three or more heads?

For the final three questions, you can instead assume the coin is biased so that it comes up heads with probability $p = \frac{2}{3}$.

(g) What is the chance of observing the outcome HHHTT? What about HHHHT?

(h) What about the chance of at least one head?

(i) What about the chance of ≥ 3 heads?

3 Past Probabilified

Note 13

In this question we review some of the past CS70 topics, and look at them probabilistically. For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}[\omega]$.

iii. Compute $\mathbb{P}[E_1]$.

iv. Compute $\mathbb{P}[E_2]$.

(a) Fix a prime $q > 2$, and uniformly sample twice with replacement from $\{0, \ldots, q-1\}$ (assume we have two $\{0, \ldots, q-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other modulo q.

Let $E_1 =$ The resulting product is 0.

Let $E_2 =$ The product is $(q-1)/2$.

(b) Make a graph on v vertices by sampling uniformly at random from all possible edges. Here, assume for each edge we flip a fair coin; if it comes up heads, we include the edge in the graph, and otherwise we exclude that edge from the graph.

Let $E_1 =$ The graph is complete.

Let $E_2 =$ vertex v_1 has degree d.

(c) Create a random stable matching instance by having each person's preference list be a uniformly random permutation of the opposite entity's list (make the preference list for each individual job and each individual candidate a random permutation of the opposite entity's list). Finally, create a uniformly random pairing by matching jobs and candidates up uniformly at random (note that in this pairing, (1) a candidate cannot be matched with two different jobs, and a job cannot be matched with two different candidates (2) the pairing does not have to be stable).

Let E_1 = All jobs have distinct favorite candidates.

Let E_2 = The resulting pairing is the candidate-optimal stable pairing.

4 Cliques in Random Graphs

Note 13

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \ldots, v_\ell\}$ and $V_2 = \{w_1, \ldots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a k-clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$.

5 PIE Extended

Note 14

One interesting result yielded by the Principle of Inclusion and Exclusion (PIE) is that for any events $A_1, A_2, \ldots, A_n$ in some probability space,

$$\sum_{i=1}^{n} \mathbb{P}[A_i] - \sum_{i<j\leq n} \mathbb{P}\left[A_i \cap A_j\right] + \sum_{i<j<k\leq n} \mathbb{P}\left[A_i \cap A_j \cap A_k\right] - \cdots + (-1)^{n-1}\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] \geq 0$$

(Note the LHS is equal to $\mathbb{P}[\bigcup_{i=1}^{n} A_i]$ by PIE, and probability is nonnegative).

Prove that for any events $A_1, A_2, \ldots, A_n$ in some probability space,

$$\sum_{i=1}^{n} \mathbb{P}[A_i] - 2\sum_{i<j\leq n} \mathbb{P}\left[A_i \cap A_j\right] + 4\sum_{i<j<k\leq n} \mathbb{P}\left[A_i \cap A_j \cap A_k\right] - \cdots + (-2)^{n-1}\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] \geq 0$$

(Hint: consider defining an event B to represent "an odd number of $A_1, \ldots, A_n$ occur")

6 Independent Complements

Note 14

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.