Due: Saturday, 3/25, 4:00 PM
Grace period until Saturday, 3/25, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Monty Hall's Revenge

Note 13
Note 14

Due to a quirk of the television studio's recruitment process, Monty Hall has ended up drawing all the contestants for his game show from among the ranks of former CS70 students. Unfortunately for Monty, the former students' amazing probability skills have made his cars-and-goats gimmick unprofitable for the studio. Monty decides to up the stakes by asking his contestants to generalise to three new situations with a variable number of doors, goats, and cars:

(a) There are n doors for some $n > 2$. One has a car behind it, and the remaining $n-1$ have goats. As in the ordinary Monty Hall problem, Monty will reveal one door with a goat behind it after you make your first selection. How would switching affect the odds that you select the car? (Compute the probability of winning in both scenarios, and compare the results.)

(Hint: Think about the size of the sample space for the experiment where you *always* switch. How many of those outcomes are favorable?)

(b) Again there are $n > 2$ doors, one with a car and $n-1$ with goats, but this time Monty will reveal $n-2$ doors with goats behind them instead of just one. How does switching affect the odds of winning in this modified scenario?

(c) Finally, imagine there are $k < n-1$ cars and $n-k$ goats behind the $n > 2$ doors. After you make your first pick, Monty will reveal $j < n-k$ doors with goats. What values of j,k maximize the relative improvement in your odds of winning if you choose to switch? (i.e. what j,k maximizes the ratio between your odds of winning when you switch, and your odds of winning when you do not switch?)

2 Man Speaks Truth

Note 14

Consider a man who speaks the truth 3 out of 4 times.

(a) Suppose the man flips a biased coin that comes up heads $1/3$ of the time, and reports that it is heads.

   (i) What is the probability that the coin actually landed on heads?

   (ii) Unconvinced, you ask him if he just lied to you, to which he replies "no". What is the probability now that the coin actually landed on heads?

   (iii) Did the probability go up, go down, or stay the same with this new information? Explain in words why this should be the case.

(b) Suppose the man rolls a fair 6-sided die. When you ask him if the die came up with a 6, he answers "yes".

   (i) What is the probability that the die actually came up with a 6?

   (ii) Skeptical, you also ask him whether the die came up with a 1, to which he replies "yes". What is the probability now that the die actually came up with a 6?

   (iii) Did the probability go up, go down, or stay the same with this new information? Explain in words why this should be the case.

3 Mario's Coins

Note 14

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

4 Symmetric Marbles

Note 14

A bag contains 4 red marbles and 4 blue marbles. Rachel and Brooke play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Rachel wins if there are more red than blue marbles, and Brooke wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?

(b) What is the probability that Rachel wins the game?

(c) Given that Rachel wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Rachel wins if the third marble is red, and Brooke wins if the third marble is blue. All other rules stay the same.

(d) What is the probability that the third marble is red?

(e) Given that there are k red marbles among the four drawn, where $0 \leq k \leq 4$, what is the probability that the third marble is red? Answer in terms of k.

(f) Given that the third marble is red, what is the probability that Rachel wins the game?

5 Cookie Jars

Note 15

You have two jars of cookies, each of which starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$) and eat one cookie from that jar. One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X be the random variable representing the number of remaining cookies in non-empty jar at that time. What is the distribution of X?

6 Testing Model Planes

Note 15

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying any remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.