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Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d + 1$ pts.

Proof sketches.

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Two polynomials: $P(x), Q(x)$, $P(x) - Q(x)$ has too many roots.

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Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

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Must prove **Roots fact.**

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Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

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In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder r.

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In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder r.

That is, $P(x) = (x - a)Q(x) + r$

Only d roots.

Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0:
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In the rationals, the precision blows up, where in modular arithmetic, it does not.

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Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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Roubustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.

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1. Choose $a_0 = s$, and randomly $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

Secret Sharing

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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3 kids hand out 3 points. Any two know the line.

Secret Sharing.

n people, k is enough.

(A) The modulus needs to be at least $n+1$.
(B) The modulus needs to be at least k.
(C) Use degree k polynomial, hand out n points.
(D) Use degree n polynomial, hand out k points.
(E) Use degree $k-1$ polynomial, hand out n points.
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Erasure Codes.

Satellite

GPS device

Erasure Codes.

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3 packet message.

GPS device

Erasure Codes.

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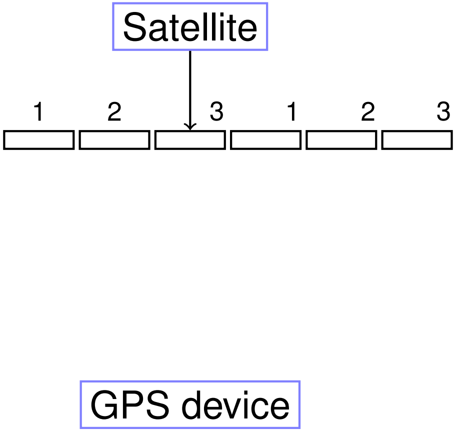
3 packet message.

Lose 3 out 6 packets.

GPS device

Erasure Codes.



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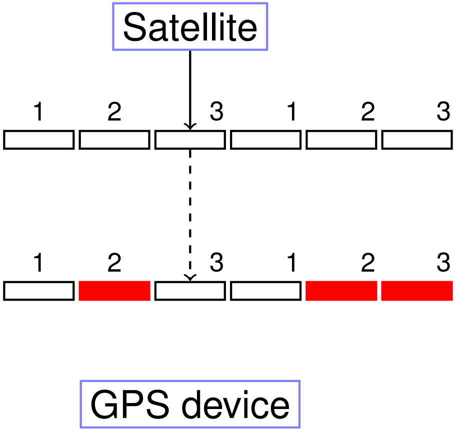

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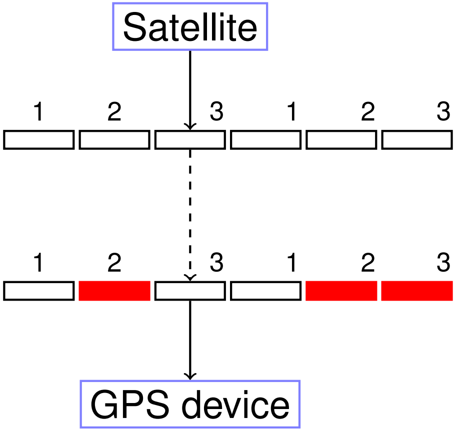

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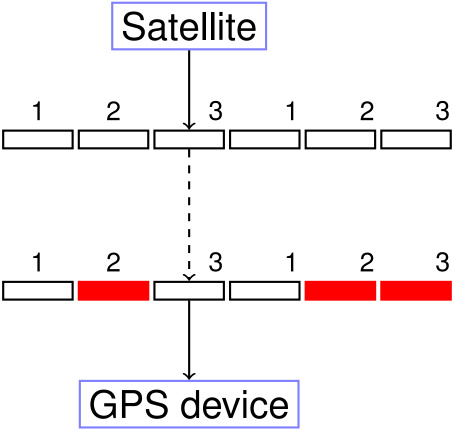

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Solution Idea.

n packet message, channel that loses *k* packets.

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Use polynomials.

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Problem: Want to send a message with n packets.

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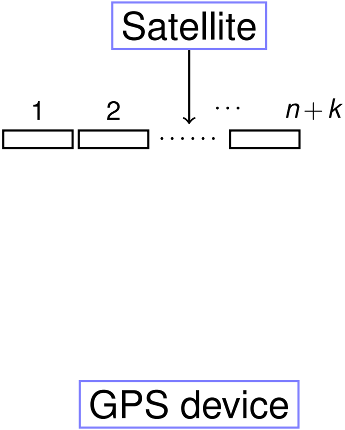

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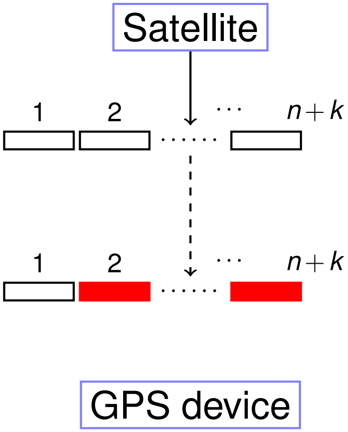

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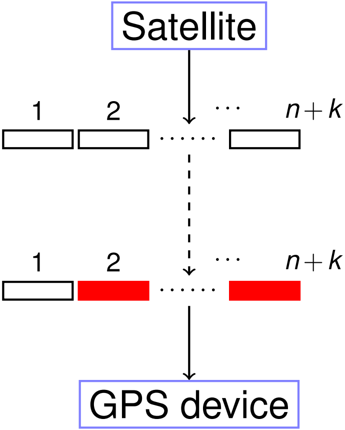

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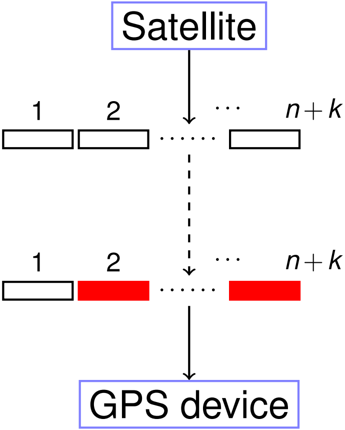

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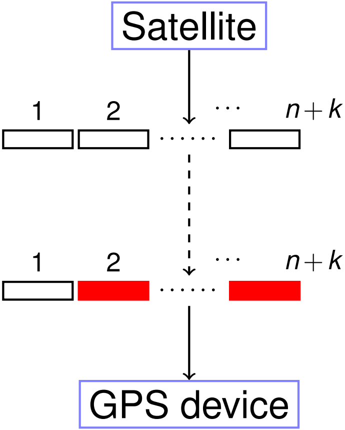

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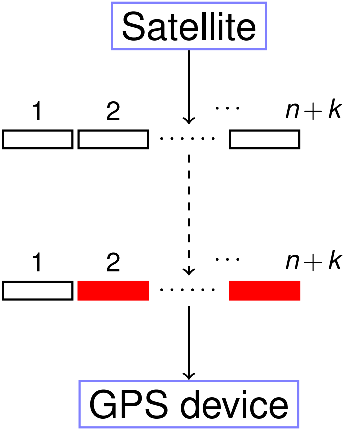

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$6a_1 + 3a_0 = 2 \pmod 7$, $5a_1 + 4a_0 = 0 \pmod 7$

$a_1 = 2a_0$. $a_0 = 2 \pmod 7$ $a_1 = 4 \pmod 7$ $a_2 = 2 \pmod 7$

Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

Modulo 7 to accommodate at least 6 packets.

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Packets: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

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Notice that packets contain “x-values”.

Bad reception!

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Lagrange or linear equations.

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Channeling Sahai

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You want to encode a secret consisting of 1,4,4.

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through a noisy channel that loses 3 packets.

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Send n packets b-bit packets, with k errors.

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Send n packets b-bit packets, with k errors.
Modulus should be larger than $n+k$ and also larger than 2^b.

Polynomials.

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- ..give Secret Sharing.

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Additional Challenge: Finding which packets are corrupt.

Error Correction

Satellite

GPS device

Error Correction

Satellite

3 packet message.

GPS device

Error Correction

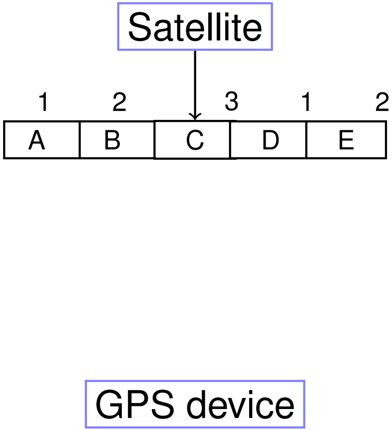
Satellite

3 packet message.

Corrupts 1 packets.

GPS device

Error Correction



3 packet message. Send 5.

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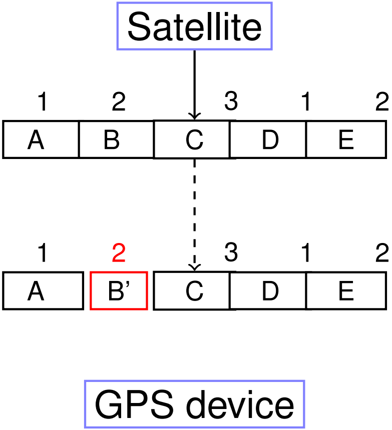

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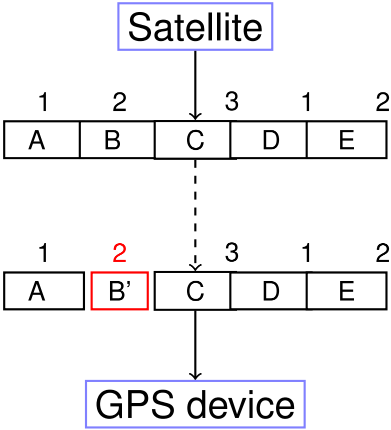

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The Scheme.

Problem: Communicate n packets $m_1, \ldots, m_n$
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(1) $P(i) = R(i)$ for at least $n+k$ points i,

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Properties:
(1) $P(i) = R(i)$ for at least $n+k$ points i,
(2) $P(x)$ is unique degree $n-1$ polynomial
that contains $\geq n+k$ received points.

Properties: proof.

$P(x)$: degree $n-1$ polynomial.

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(2) Degree $n-1$ polynomial $Q(x)$ consistent with $n+k$ points.
$Q(x)$ agrees with $R(i)$, $n+k$ times.
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Total points contained by both: $2n+2k$.

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$\implies$ $Q(i) = P(i)$ at n points.

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$P(i) = R(i)$ for $n + k = 3 + 1 = 4$ points.

Slow solution.

Brute Force:
For each subset of $n+k$ points

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Reconstructs $P(x)$ and only $P(x)$!!

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Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

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Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

All equations..

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\begin{aligned}
p_2 + p_1 + p_0 &\equiv 3 \pmod 7 \\
4p_2 + 2p_1 + p_0 &\equiv 1 \pmod 7 \\
2p_2 + 3p_1 + p_0 &\equiv 6 \pmod 7 \\
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Assume point 1 is wrong and solve..

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&. \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod{p}
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In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n + 2k)$.

$$
\begin{aligned}
p_{n-1} + \cdots p_0 &\equiv R(1) \pmod{p} \\
p_{n-1}2^{n-1} + \cdots p_0 &\equiv R(2) \pmod{p} \\
&\cdot \\
p_{n-1}i^{n-1} + \cdots p_0 &\equiv R(i) \pmod{p} \\
&\cdot \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod{p}
\end{aligned}
$$

Error!!

In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n + 2k)$.

$$\begin{aligned}
p_{n-1} + \cdots p_0 &\equiv R(1) \pmod{p} \\
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p_{n-1}i^{n-1} + \cdots p_0 &\equiv R(i) \pmod{p} \\
&. \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod{p}
\end{aligned}$$

Error!! .... Where???

In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n + 2k)$.

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\begin{aligned}
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\end{aligned}
$$

Error!! .... Where???
Could be anywhere!!!

In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n + 2k)$.

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&\cdot \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod p
\end{aligned}
$$

Error!! .... Where???

Could be anywhere!!! ...so try everywhere.

In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n+2k)$.

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&\cdot \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod p
\end{aligned}
$$

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.
Runtime: $\binom{n+2k}{k}$ possibilitities.

In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n + 2k)$.

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\end{aligned}
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Error!! .... Where???
Could be anywhere!!! ...so try everywhere.
Runtime: $\binom{n+2k}{k}$ possibilitities.

Something like $(n/k)^k$ ...Exponential in k!.

In general..

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\end{aligned}$$

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.
Runtime: $\binom{n+2k}{k}$ possibilitities.

Something like $(n/k)^k$ ...Exponential in k!.

How do we find where the bad packets are efficiently?!?!?!

Ditty...

Oh where, Oh where

Ditty...

Oh where, Oh where
has my little dog gone?

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone..

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?
Oh where, oh where do they not fit.

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?
Oh where, oh where do they not fit.

With the polynomial well put

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?
Oh where, oh where do they not fit.

With the polynomial well put
But the channel a bit wrong

Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?
Oh where, oh where do they not fit.

With the polynomial well put
But the channel a bit wrong
Where, oh where do we look?

Where oh where can my bad packets be?

$$(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}$$

Where oh where can my bad packets be?

$$
\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
(p_{n-1} 2^{n-1} + \cdots p_0) &\equiv R(2) \pmod{p} \\
&\vdots \\
(p_{n-1} (m)^{n-1} + \cdots p_0) &\equiv R(n+2k) \pmod{p}
\end{aligned}
$$

Where oh where can my bad packets be?

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Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.

Where oh where can my bad packets be?

$$(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}$$
$$(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}$$
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$$(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!!

Where oh where can my bad packets be?

$$(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}$$

$$(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}$$

$$\vdots$$

$$(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.

Where oh where can my bad packets be?

$$
\begin{aligned}
& (p_{n-1} + \cdots p_0) &\equiv\quad& R(1) && \pmod{p} \\
\mathbf{0} \times \quad & (p_{n-1}2^{n-1} + \cdots p_0) &\equiv\quad& R(2) && \pmod{p} \\
& & \vdots & & & \\
& (p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv\quad& R(n+2k) && \pmod{p}
\end{aligned}
$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

Where oh where can my bad packets be?

$$\begin{aligned}(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p}\\ (p_{n-1}2^{n-1} + \cdots p_0) &\equiv R(2) \pmod{p}\\ &\vdots\\ (p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv R(n+2k) \pmod{p}\end{aligned}$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0?

Where oh where can my bad packets be?

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\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
(p_{n-1} 2^{n-1} + \cdots p_0) &\equiv R(2) \pmod{p} \\
&\vdots \\
(p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv R(n+2k) \pmod{p}
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But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!!

Where oh where can my bad packets be?

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\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
(p_{n-1} 2^{n-1} + \cdots p_0) &\equiv R(2) \pmod{p} \\
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Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don't know.

Where oh where can my bad packets be?

$$
\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
(p_{n-1} 2^{n-1} + \cdots p_0) &\equiv R(2) \pmod{p} \\
&\vdots \\
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Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don't know. But can find!

Where oh where can my bad packets be?

$$\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
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But which equations should we multiply by 0? Where oh where...??

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Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Where oh where can my bad packets be?

$$\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
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But which equations should we multiply by 0? Where oh where...??

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Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Error locator polynomial: $E(x) = (x - e_1)$

Where oh where can my bad packets be?

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\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
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Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don't know. But can find!

Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Error locator polynomial: $E(x) = (x - e_1)(x - e_2)$

Where oh where can my bad packets be?

$$\begin{aligned}(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p}\\ (p_{n-1}2^{n-1} + \cdots p_0) &\equiv R(2) \pmod{p}\\ &\vdots\\ (p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv R(n+2k) \pmod{p}\end{aligned}$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
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But which equations should we multiply by 0? Where oh where...??

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Error locator polynomial: $E(x) = (x - e_1)(x - e_2)\ldots$

Where oh where can my bad packets be?

$$
\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) && \pmod{p} \\
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&\vdots \\
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Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Error locator polynomial: $E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)$.

Where oh where can my bad packets be?

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(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
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Error locator polynomial: $E(x) = (x - e_1)(x - e_2) \ldots (x - e_k)$.

$E(i) = 0$ if and only if $e_j = i$ for some j

Where oh where can my bad packets be?

$$\begin{aligned} E(1)(p_{n-1} + \cdots p_0) &\equiv R(1)E(1) \pmod{p} \\ E(2)(p_{n-1}2^{n-1} + \cdots p_0) &\equiv R(2)E(2) \pmod{p} \\ &\vdots \\ E(m)(p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv R(n+2k)E(m) \pmod{p} \end{aligned}$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don't know. But can find!

Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

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$E(i) = 0$ if and only if $e_j = i$ for some j

Multiply equations by $E(\cdot)$.

Where oh where can my bad packets be?

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Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don't know. But can find!

Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Error locator polynomial: $E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)$.

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Multiply equations by $E(\cdot)$. (Above E(x) = (x-2).)

Where oh where can my bad packets be?

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Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don't know. But can find!

Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Error locator polynomial: $E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)$.

$E(i) = 0$ if and only if $e_j = i$ for some j

Multiply equations by $E(\cdot)$. (Above E(x) = (x-2).)

All equations satisfied!!

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$
\begin{aligned}
(p_2 + p_1 + p_0) &\equiv (3) \pmod 7 \\
(4p_2 + 2p_1 + p_0) &\equiv (1) \pmod 7 \\
(2p_2 + 3p_1 + p_0) &\equiv (6) \pmod 7 \\
(2p_2 + 4p_1 + p_0) &\equiv (0) \pmod 7 \\
(4p_2 + 5p_1 + p_0) &\equiv (3) \pmod 7
\end{aligned}
$$

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

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(2p_2 + 4p_1 + p_0) &\equiv (0) \pmod 7 \\
(4p_2 + 5p_1 + p_0) &\equiv (3) \pmod 7
\end{aligned}$$

Error locator polynomial: $(x - 2)$.

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$\begin{aligned}
(1-2)(p_2 + p_1 + p_0) &\equiv (3)(1-2) \pmod 7 \\
(2-2)(4p_2 + 2p_1 + p_0) &\equiv (1)(2-2) \pmod 7 \\
(3-2)(2p_2 + 3p_1 + p_0) &\equiv (6)(3-2) \pmod 7 \\
(4-2)(2p_2 + 4p_1 + p_0) &\equiv (0)(4-2) \pmod 7 \\
(5-2)(4p_2 + 5p_1 + p_0) &\equiv (3)(5-2) \pmod 7
\end{aligned}$$

Error locator polynomial: $(x - 2)$.

Multiply equation i by $(i - 2)$.

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$
\begin{aligned}
(1-2)(p_2 + p_1 + p_0) &\equiv (3)(1-2) \pmod 7 \\
(2-2)(4p_2 + 2p_1 + p_0) &\equiv (1)(2-2) \pmod 7 \\
(3-2)(2p_2 + 3p_1 + p_0) &\equiv (6)(3-2) \pmod 7 \\
(4-2)(2p_2 + 4p_1 + p_0) &\equiv (0)(4-2) \pmod 7 \\
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Error locator polynomial: $(x - 2)$.

Multiply equation i by $(i - 2)$. All equations satisfied!

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

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(1-2)(p_2 + p_1 + p_0) &\equiv (3)(1-2) \pmod 7\\
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4 unknowns (p_0, p_1, p_2 and e), 5 nonlinear equations.

..turn their heads each day,

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\begin{aligned}
(p_{n-1} + \cdots p_0) &\equiv R(1) \pmod{p} \\
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(p_{n-1} i^{n-1} + \cdots p_0) &\equiv R(i) \pmod{p} \\
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(p_{n-1}(n+2k)^{n-1} + \cdots p_0) &\equiv R(m) \pmod{p}
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and linear in a_i and coefficients of $E(x)$!

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For all points $1, \ldots, i, n+2k = m$,

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$$a_3 + a_2 + a_1 + a_0 \equiv 3(1 - b_0) \pmod 7$$

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$Q(x) = E(x)P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$E(x) = x - b_0$

$Q(i) = R(i)E(i)$.

$$
\begin{aligned}
a_3 + a_2 + a_1 + a_0 &\equiv 3(1 - b_0) \pmod 7 \\
a_3 + 4a_2 + 2a_1 + a_0 &\equiv 1(2 - b_0) \pmod 7
\end{aligned}
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$Q(x) = x^3 + 6x^2 + 6x + 5$.

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$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

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Example: finishing up.

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```
          ------------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                  1 x^2
         ------------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                1 x^2
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                1 x^2 + 1 x
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
                1 x^2 - 2 x
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                  1 x^2 + 1 x
          -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                  1 x^2 + 6 x + 5
                  1 x^2 - 2 x
                  ---------------
                            x + 5
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                 1 x^2 + 1 x + 1
        -----------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
        x^3  - 2 x^2
        ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                  1 x^2 + 1 x + 1
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
                1 x^2 - 2 x
                ---------------
                          x + 5
                          x - 2
                          -----
                              0
```

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$Q(x) = x^3 + 6x^2 + 6x + 5.$
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```
                 1 x^2 + 1 x + 1
          -----------------
x - 2 )   x^3  + 6 x^2 + 6 x + 5
          x^3  - 2 x^2
          ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1$

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                 1 x^2 + 1 x + 1
          -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                 1 x^2 + 1 x + 1
          -----------------
  x - 2 ) x^3  + 6 x^2 + 6 x + 5
          x^3  - 2 x^2
          ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$?

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
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```
                  1 x^2 + 1 x + 1
            -----------------
    x - 2 ) x^3  + 6 x^2 + 6 x + 5
            x^3  - 2 x^2
            ----------
                  1 x^2 + 6 x + 5
                  1 x^2 - 2 x
                  ---------------
                            x + 5
                            x - 2
                            -----
                                0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$? 1

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$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                1 x^2 + 1 x + 1
           -----------------
    x - 2 ) x^3  + 6 x^2 + 6 x + 5
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            ----------
                1 x^2 + 6 x + 5
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                ---------------
                          x + 5
                          x - 2
                          -----
                              0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$? 1
Except at $x = 2$?

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                  1 x^2 + 1 x + 1
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                  1 x^2 + 6 x + 5
                  1 x^2 - 2 x
                  ---------------
                            x + 5
                            x - 2
                            -----
                                0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$? 1
Except at $x = 2$? Hole there?

Error Correction: Berlekamp-Welsh

Message: $m_1, \ldots, m_n$.
Sender:

1. Form degree $n-1$ polynomial $P(x)$ where $P(i) = m_i$.
2. Send $P(1), \ldots, P(n+2k)$.

Receiver:

1. Receive $R(1), \ldots, R(n+2k)$.
2. Solve $n+2k$ equations, $Q(i) = E(i)R(i)$ to find $Q(x) = E(x)P(x)$ and $E(x)$.
3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1), \ldots, P(n)$.

Check your undersanding.

You have error locator polynomial!

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Check your undersanding.

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Where oh where have my packets gone wrong?

Factor?

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values?

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Efficiency?

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Efficiency? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Efficiency? Sure. Only $n+2k$ values.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Efficiency? Sure. Only $n+2k$ values.
See where it is 0.

Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

Existence: there is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \tag{1}$$

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Proof:

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Proof:
We claim

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Proof:
We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \tag{2}$$

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$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
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$E(x)$ and $E'(x)$ have at most k zeros each.

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Can cross divide at n points.

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Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

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$$Q(i) = R(i)E(i)$$
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for $i \in \{1, \ldots n+2k\}$.

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If $E(i) = 0$, then $Q(i) = 0$.

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If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

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Cross multiplying gives equality in fact for these points.

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for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Cross multiplying gives equality in fact for these points. □

Points to polynomials, have to deal with zeros!

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \dots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
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Cross multiplying gives equality in fact for these points. □

Points to polynomials, have to deal with zeros!

Example: dealing with $\frac{x-2}{x-2}$ at $x = 2$.

Yaay!!

Berlekamp-Welsh algorithm decodes correctly when k errors!

Poll

Say you sent a message of length 4, encoded as $P(x)$ where one sends packets $P(1), ...P(8)$.

You recieve packets $R(1), ...R(8)$.

Packets 1 and 4 are corrupted.

(A) $R(1) \neq P(1)$
(B) The degree of $P(x)E(x) = 3+2 = 5$.
(C) The degree of $E(x)$ is 2.
(D) The number of coefficients of $P(x)$ is 4.
(E) The number of coefficients of $P(x)Q(x)$ is 6.

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(E) is false.

(A) $E(x) = (x-1)(x-4)$
(B) The number of coefficents in $E(x)$ is 2.
(C) The number of unknown coefficents in $E(x)$ is 2.
(D) $E(x) = (x-1)(x-2)$
(E) $R(4) \neq P(4)$
(F) The degree of $R(x)$ is 5.

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(A), (C), (E). (F) doesn't type check!

Summary. Error Correction.

Communicate n packets, with k erasures.

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How many packets?

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Of degree?

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Reed-Solomon codes.

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Reed-Solomon codes. Welsh-Berlekamp Decoding.

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Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!

Cool.

Really Cool!