Today.

Finish up counting.

Thoughts on content...

...and midterm.

Sampling...

Sample k items out of n

Without replacement:
Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
Order does not matter:
Second Rule: divide by number of orders – "k!"
$\implies \frac{n!}{(n-k)!k!}$.
"n choose k"

With Replacement.
Order matters: $n \times n \times \ldots n = n^k$
Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
Different number of unordered elts map to each unordered elt.

Unordered elt: 1,2,3 3! ordered elts map to it.
Unordered elt: 1,2,2 $\frac{3!}{2!}$ ordered elts map to it.

How do we deal with this mess??

Some Practice.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways for second, 1 for last.

$\implies 3 \times 2 \times 1 = 3!$ orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A!

total orderings of 7 letters. 7!
total "extra counts" or orderings of three A's? 3!

Total orderings? $\frac{7!}{3!}$

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.
11 letters total.
11! ordered objects.
$4! \times 4! \times 2!$ ordered objects per "unordered object"

$\implies \frac{11!}{4!4!2!}$.

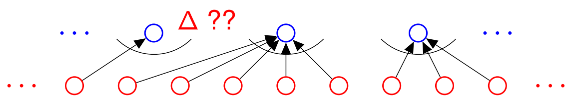
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice(2^5), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: (B,B,B,B,B).

4 for Bob and 1 for Alice:
5 ordered sets: (A,B,B,B,B) ; (B,A,B,B,B); ...

"Sorted" way to specify, first Alice's dollars, then Bob's.
(B,B,B,B,B): 1: (B,B,B,B,B)
(A,B,B,B,B): 5: (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
(A,A,B,B,B): $\binom{5}{2}$;(A,A,B,B,B),(A,B,A,B,B),(A,B,B,A,B),...
and so on.



Second rule of counting is no good here!

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter (sometimes) can divide...

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.
pause Bijection!

Sample k times from n objects with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A,A,A,B,E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$- - - - - - -$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star | \star \star \star \star |$.
Bars in second and seventh position.

$\binom{7}{2}$ ways to do so and
$\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars.

Ways to add up n numbers to sum to k? or

" k from n with replacement where order doesn't matter."

In general, k stars $n-1$ bars.

$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn't matter.

Counting basics.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter divide..when possible.

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Bijection: sums to 'k' $\rightarrow$ stars and bars.

$S = \{(n_1,n_2,n_3) : n_1+n_2+n_3 = 5\}$

$T = \{s \in \{'|','\star'\} : |s| = 7, \text{number of bars in s} = 2\}$

$f((n_1,n_2,n_3)) = \star^{n_1} \quad '|' \quad \star^{n_2} \quad '|' \quad \star^{n_3}$

Bijection:
argument: unique (n_1,n_2,n_3) from any s.

$|S| = |T| = \binom{7}{2}$.

Stars and Bars Poll

Mark whats correct.
(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$
(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$
(C) ways to split 5 dollars among 3: $\binom{7}{5}$
(D) ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$

All correct.

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.

Poll

Mark whats correct.

k Balls in n bins.

dis == distinguishiable
unique = one ball in each bin.

(A) dis $=> n^k$
(B) dis,unique $=> n!/(n-k)!$
(C) indis, unique $=> \binom{n}{k}$
(D) dis, $=> n!/(n-k)!$
(E) indis, $=> \binom{n+k-1}{k-1}$
(F) dis,unique $=> \binom{n}{k}$

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers "exclusive" or One Joker "exclusive" or Two Jokers

$$\binom{52}{5}+\binom{52}{4}+\binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5}+2*\binom{52}{4}+\binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5}=\binom{52}{5}+2*\binom{52}{4}+\binom{52}{3}$.

Algebraic Proof: Why? Just why? Especially on Tuesday!
Already have a **combinatorial proof.** □

Combinatorial Proofs.

Theorem: $\binom{n}{k}=\binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

□

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Row n: coefficients of $(1+x)^n=(1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms: choose 1 or x from each term $(1+b)$.

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k $\binom{n}{k}$: choose k terms with x in product.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$

Pascal's rule $\implies \binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$.
Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?
Chose first element, need $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elts.
$\implies \binom{n}{k}$
Sum Rule: size of union of disjoint sets of objects.
Without and with first element $\rightarrow$ disjoint.

So, $\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$. □

Combinatorial Proof.

Theorem: $\binom{n}{k}=\binom{n-1}{k-1}+\cdots+\binom{k-1}{k-1}$.

Proof: Consider size k subset where i is the first element chosen.

$$\{1,\ldots,i,\underline{\ldots,n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

Add them up to get the total number of subsets of size k
which is also $\binom{n+1}{k}$.

□

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1,\ldots,n\}$?
Construct a subset with sequence of n choices:
element i **is in** or **is not** in the subset: 2 poss.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

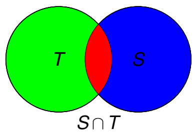
How many subsets of $\{1,\ldots,n\}$?
$\binom{n}{i}$ ways to choose i elts of $\{1,\ldots,n\}$.
Sum over i to get total number of subsets..which is also 2^n. $\square$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 1, 2, 3,...

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule:
For any S and T, $|S \cup T| = |S| + |T| - |S \cap T|$.



In T. $\implies |T|$
In S. $\implies + \ |S|$
Elements in $S \cap T$ are counted twice.
Subtract. $\implies -|S \cap T|$

$|S \cup T| = |S| + |T| - |S \cap T|$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 1, 2, 3,...

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Idea: For $n = 3$ how many times is an element counted?
Consider $x \in A_i \cap A_j$.
x counted once for $|A_i|$ and once for $|A_j|$.
x subtracted from count once for $|A_i \cap A_j|$.
Total: 2 -1 = 1.

Consider $x \in A_1 \cap A_2 \cap A_3$
x counted once in each term: $|A_1|, |A_2|, |A_3|$.
x subtracted once in terms: $|A_1 \cap A_3|$, $|A_1 \cap A_2|$, $|A_2 \cap A_3|$.
x added once in $|A_1 \cap A_2 \cap A_3|$.
Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.
3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.
$\binom{3}{3}$ for terms of form $|A_i \cap A_j \cap A_k|$.
Total: $\binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Idea: how many times is each element counted?
Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.
Counted $\binom{m}{i}$ times in ith summation.
Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m}$.

Binomial Theorem:
$(x+y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m$.
Proof: m factors in product: $(x+y)(x+y)\cdots(x+y)$.
Get a term $x^{m-i} y^i$ by choosing i factors to use for y.
are $\binom{m}{i}$ ways to choose factors where y is provided. $\square$

For $x = 1, y = -1$,
$0 = (1-1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m \binom{m}{m}$

$\implies \quad 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \cdots + (-1)^{m-1} \binom{m}{m}$.

Each element counted once!

Summary.

First Rule of counting: Objects from a sequence of choices:
n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

Second Rule of counting: If order does not matter.
Count with order: Divide number of orderings. Typically: $\binom{n}{k}$.

Stars and Bars: Sample k objects with replacement from n.
Order doesn't matter: Typically: $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

Inclusion/Exclusion: two sets of objects.
Add number of each subtract intersection of sets.
Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.
Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
RHS: Number of subsets of $n+1$ items size k.
LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.
$\binom{n}{k}$ counts subsets of $n+1$ items without first item.
Disjoint – so add!

Kolmogorov Complexity, Google, and CS70

What is the minimum I need to know (remember) to know stuff.

Radius of the earth? Distance to the sun? Population of the US?
Acceleration due to gravity on earth?
Google. Plus reference.

Syntax of pandas? Google + Stackoverflow.
Plus "how to program" and remembering a bit.

What is π?
Kolmorogorov Complexity View:
perimeter of a circle/diameter.

Calculus: what is minimum you need to know?
Depends on your skills!
Conceptualization.
Reason and understand an argument and you can generate a lot.

Calculus

What is the first half of calculus about?

The slope of a tangent line to a function at a point.

Slope is rise/run. Oh, yes: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Chain rule? Derivative of a function composition.
Intuition: composition of two linear functions?
$f(x) = ax$, $g(x) = bx$. $f(g(x)) = ab\,x$. Slope is ab.
Multiply slopes!

$(f(g(x))' = f'(\cdot)g'(\cdot)$

But...but...

For function slopes of tangent differ at different places.

So, where? $f(g(x))$
slope of f at $g(x)$ times slope of g at x.
$(f(g(x))' = f'(g(x))g'(x)$.

Proof?
Plugin definitions (e.g., $\lim_{h\to 0}\cdots$), make argument (or do derivation.)

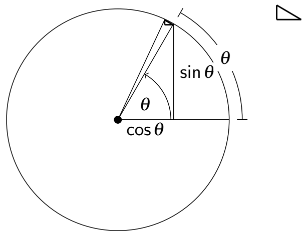
Derivative of sine?

$\sin(x)$.

What is x? An angle in radians.

Let's call it θ and do derivative of $\sin\theta$.

θ - Length of arc of unit circle



Rise. Similar triangle!!!
Rise of sine $\propto$ cosine!
Change of cosine $\propto$ -sine.

Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.

Useful?
Speed times Time is Distance.

Conceptual: Area is proportional to height.

If you change width, change in area is proportional to height.

Derivative (rate of change) of Area (Integral) under curve, is height of curve.

Pure Math: Real Analysis, Definition of area.
From the natural numbers, to rationals, to reals....

Application Math: Newton.
Velocity times Time is Distance.

$\int_a^b f(x)d(x)$ "is" area under the curve.

Arguments, reasoning.

What you know: slope, limit.
Plus: definition.
yields calculus.
Minimization, optimization, ...

Knowing how to program plus some syntax (google) gives the ability to program.

Knowing how to reason plus some definition gives calculus.

Discrete Math: basics are counting, how many, when are two sets the same size?

Probability: division.

...plus reasoning.

CS 70 : ideas.

Induction $\equiv$ every integer has a next one. Graph theory.
Number of edges is sum of degrees.
$\Delta+1$ coloring. Neighbors only take up Δ.
Connectivity plus connected components.
Eulerian paths: if you enter you can leave.
Euler's formula: tree has $v-1$ edges and 1 face plus
each extra edge makes additional face.
$v-1+(f-1)=e$

CS 70 : ideas.

Number theory.
A divisor of x and y divides $x-y$.
The remainder is always smaller than the divisor.
$\implies$ Euclid's GCD algorithm.
Multiplicative Inverse.
Fermat's theorem from function with inverse is a bijection.
Gives RSA.

Error Correction.
(Any) Two points determine a line.
(well, and d points determine a degree $d+1$-polynomials.
Cuz, factoring.
Find line by linear equations.
If a couple are wrong, then multiply them by zero, i.e., Error polynomial.