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More detail: $even + even - even = 2q + 2k - 2m = 2(q+k-m)$.

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That is 11|alternating sum of digits. □

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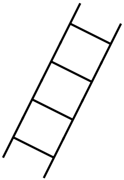
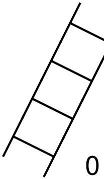


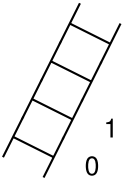
The natural numbers.

0,



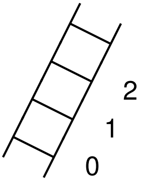
The natural numbers.

0, 1,



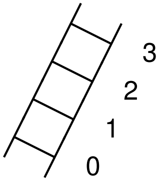
The natural numbers.

0, 1, 2,



The natural numbers.

0, 1, 2, 3,



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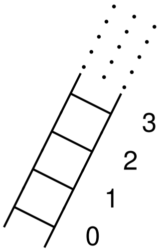

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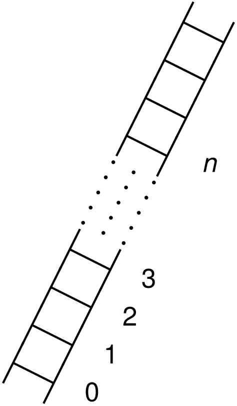

0, 1, 2, 3,

…, n,

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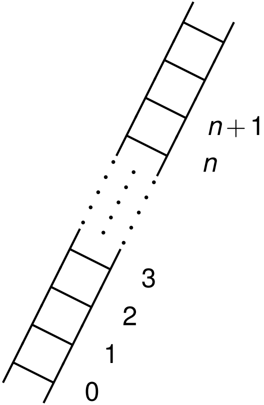

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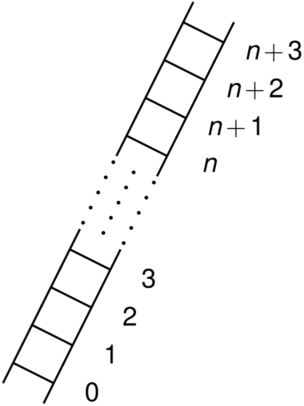

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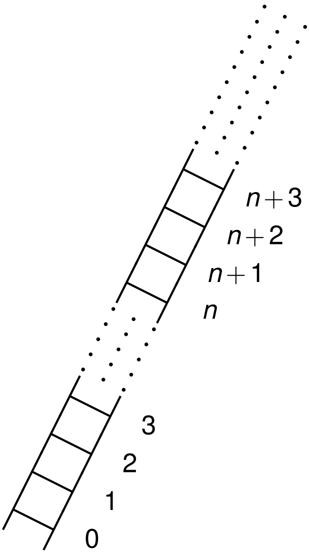

$$0, 1, 2, 3, \ldots, n, n+1, n+2, n+3, \ldots$$

A formula.

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- Prove $P(0)$.
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- Prove $P(k+1)$. "Induction Step."
- $\implies P(n)$ is true for all $n \in \mathbb{N}$.

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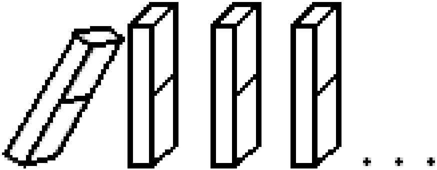
Notes visualization

Note's visualization: an infinite sequence of dominos.



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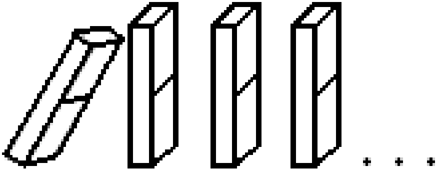

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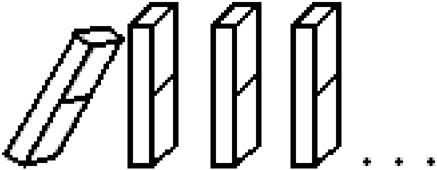

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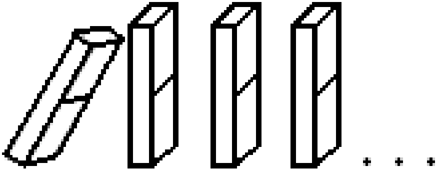

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Climb an infinite ladder?

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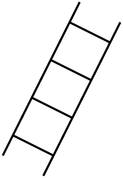

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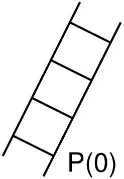

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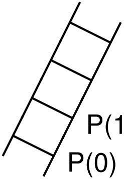

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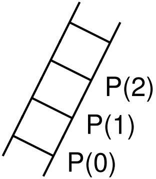

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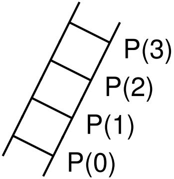

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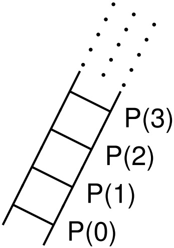

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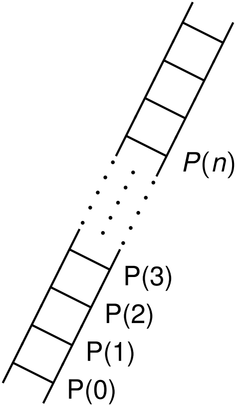

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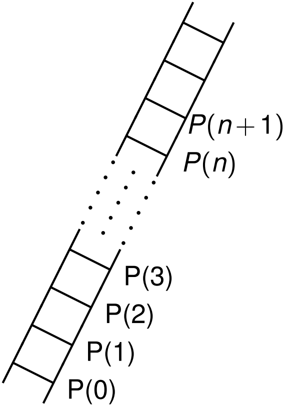

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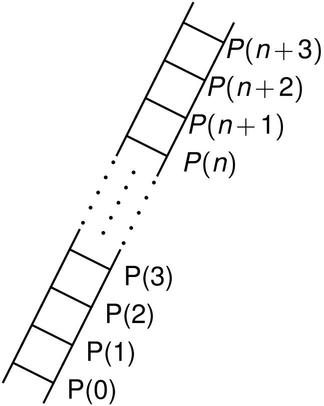

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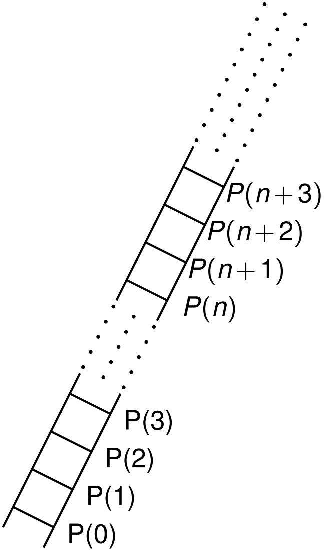

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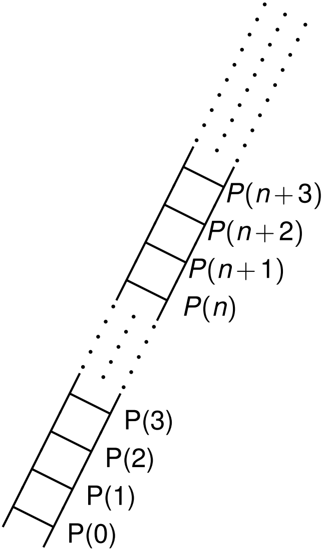

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Your favorite example of forever..

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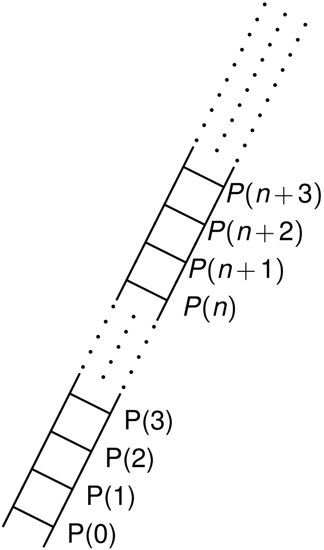

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Your favorite example of forever..or the natural numbers...

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

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Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Poll: What did Gauss use in the proof?

(A) Every natural number has a next number.

(B) The recursive leap of faith.

(C) $2^k > k$.

(D) $\forall k \in \mathbb{N}, \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

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$(k+1)^3 - (k+1)$

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We used everything above except (A). (E) is false.

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

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Proper coloring: for each line segment the regions on the two sides have different colors.1

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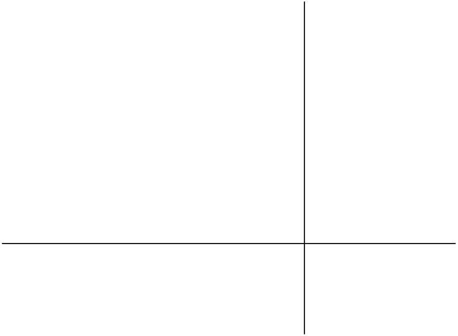

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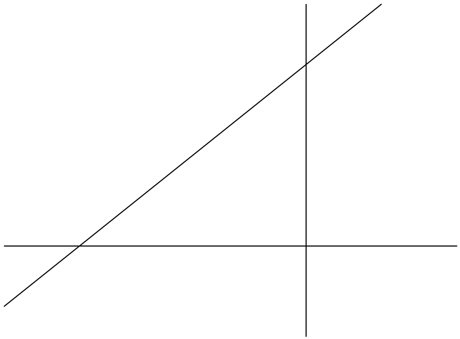

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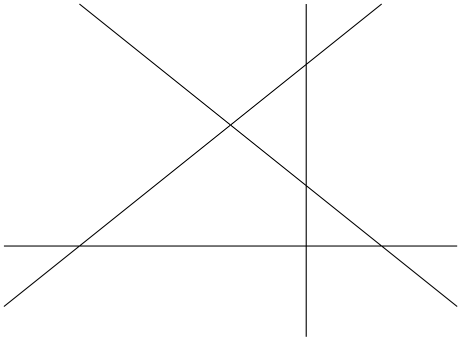

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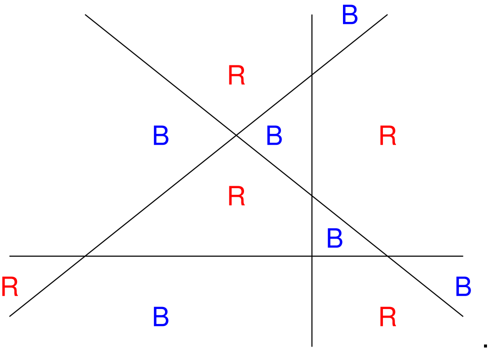

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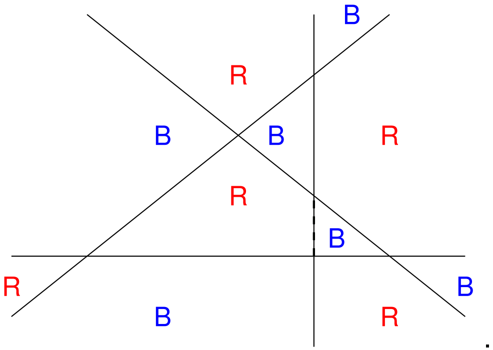

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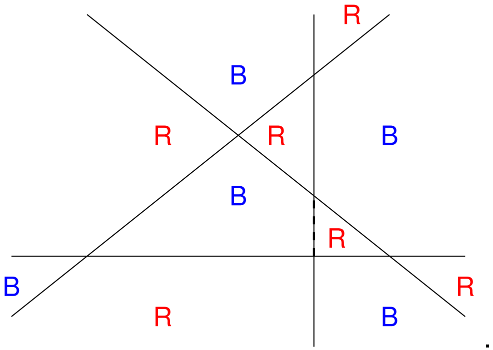

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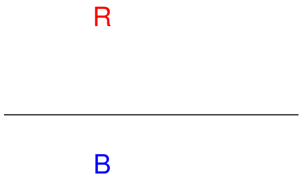

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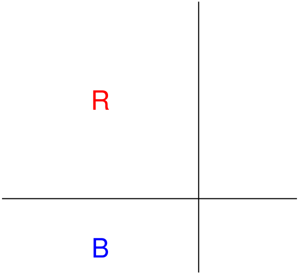

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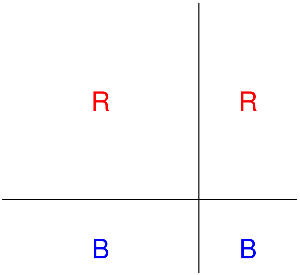

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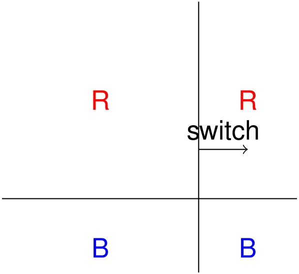

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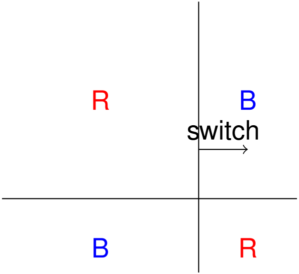

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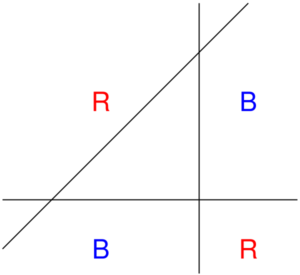

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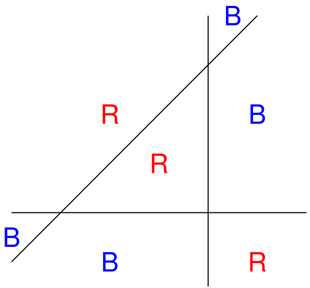

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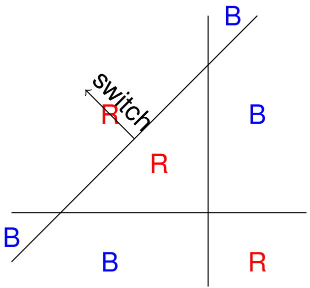

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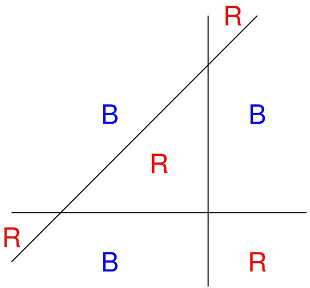

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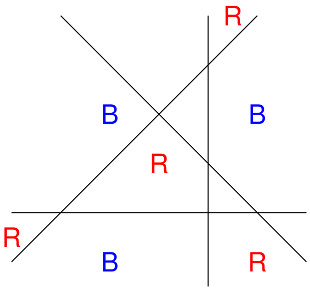

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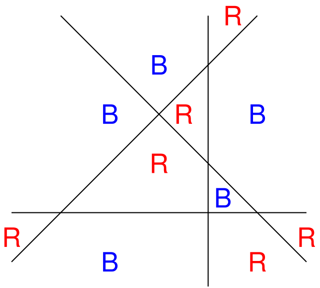

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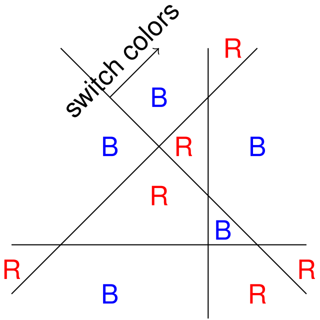

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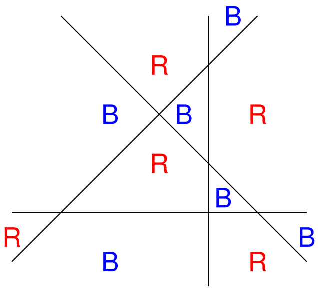

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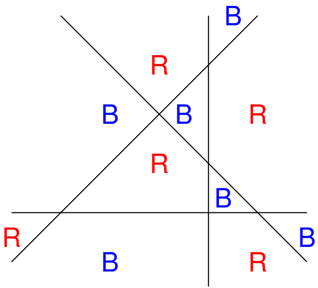

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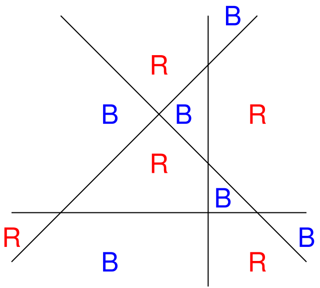

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Poll: what did we use in the proof.

(A) Switching a 2-coloring is a valid coloring.

(B) Definition of 2-coloring.

(C) Definition of adjacent.

(D) Definition of region.

(E) The four color theorem.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

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   $a^2 + 2k + 1 = k^2 + 2k + 1$
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   ... P(k+1)!

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

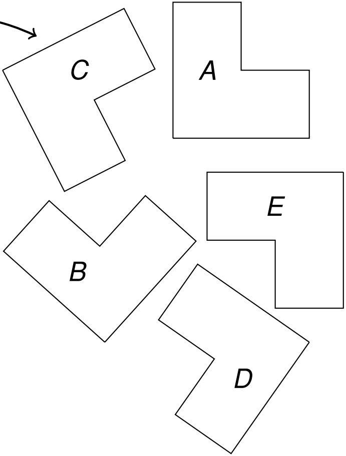
Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

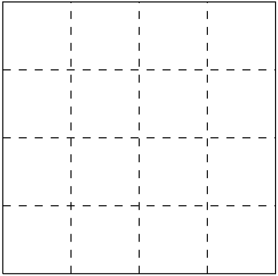
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



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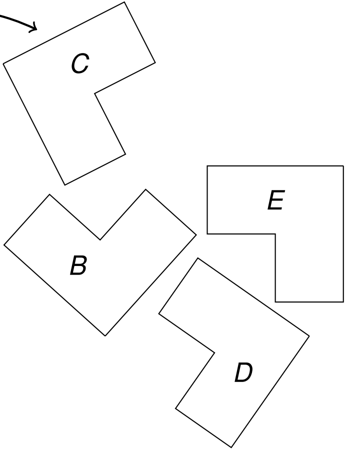

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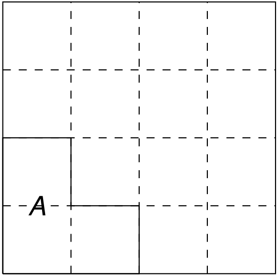

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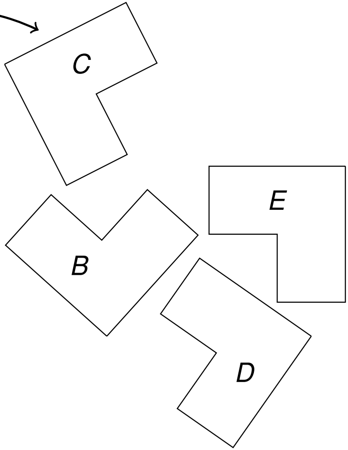

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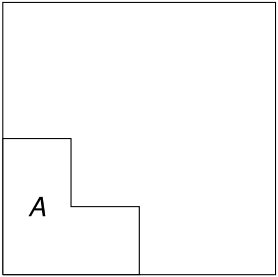

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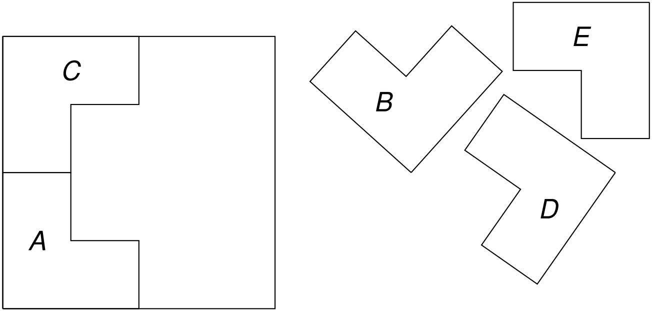

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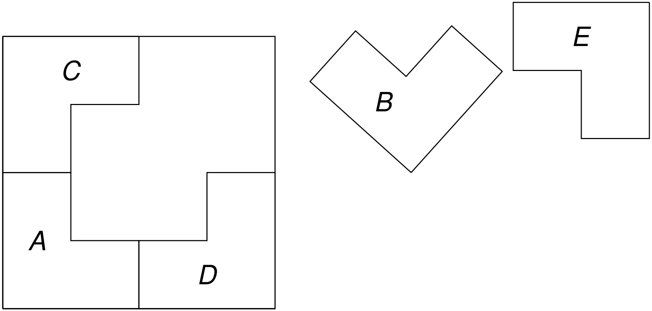

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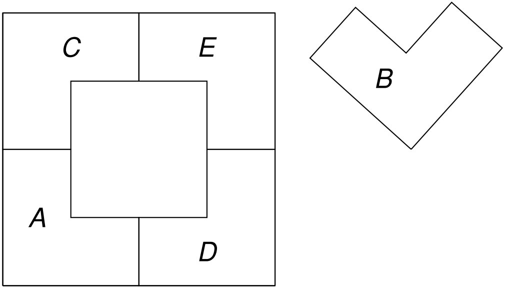

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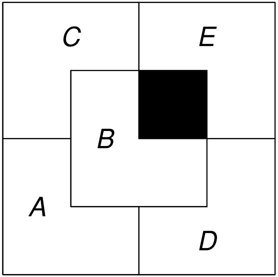

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Use these L-tiles.

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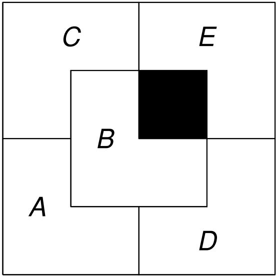
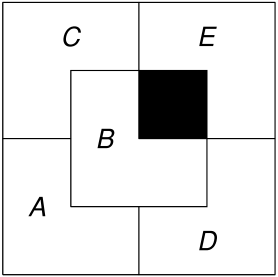


Alright!

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



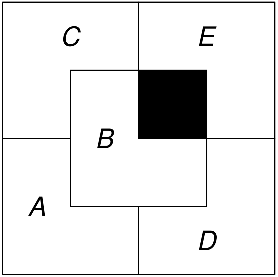
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

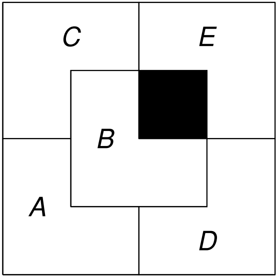
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



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Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

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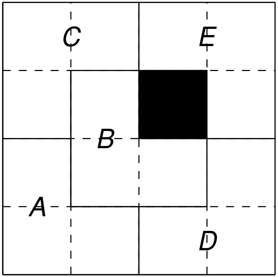

Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$.

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Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

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$$2^{2(k+1)}$$

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$$2^{2(k+1)} = 2^{2k} * 2^2$$

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$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \end{aligned}$$

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&= 4 * (3a + 1)
\end{aligned}
$$

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$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \\ &= 12a + 3 + 1 \end{aligned}$$

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a integer

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a integer $\implies$ $(4a + 1)$ is an integer.

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a integer $\Longrightarrow$ $(4a + 1)$ is an integer. $\square$

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

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Proof:

Base case: A single tile works fine.

Hole in center?

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The hole is adjacent to the center of the 2×2 square.

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Any $2^n \times 2^n$ square can be tiled with a hole at the center.

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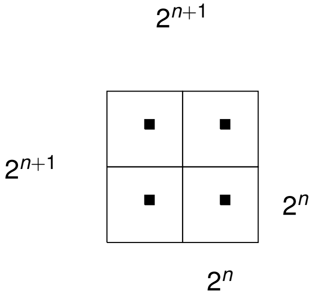

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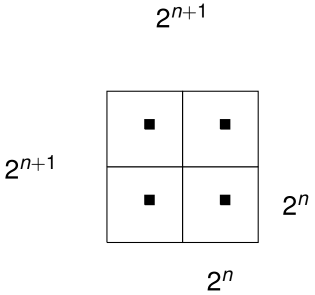

What to do now???

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

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Better theorem

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Better theorem ...better induction hypothesis!

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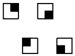
Flipping the orientation can leave hole anywhere.



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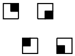

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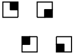

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.

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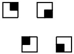

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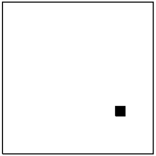

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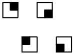

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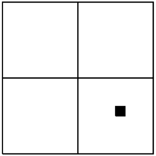

Use induction hypothesis in each.

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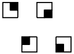

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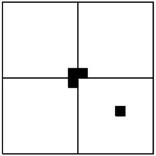

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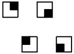
Flipping the orientation can leave hole anywhere. 

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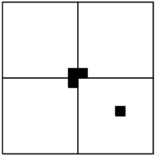

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Use L-tile and ...

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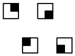

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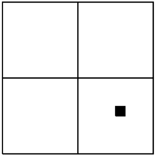

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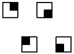
Flipping the orientation can leave hole anywhere. 

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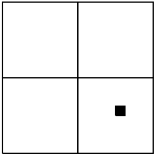

Use induction hypothesis in each.

Use L-tile and ... we are done. □

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

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$P(n) =$ "n can be written as a product of primes. "

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Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

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For example. Use reduced form: a/b and order by $a+b$.

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Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

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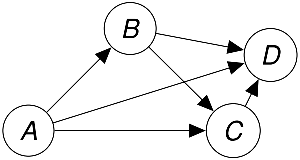

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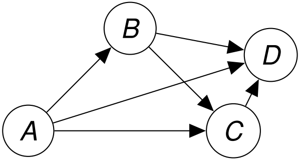

Theorem: Any tournament that has a cycle has a cycle of length 3.

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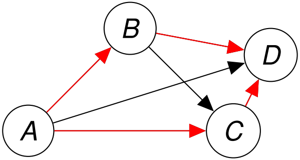

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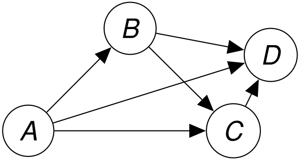

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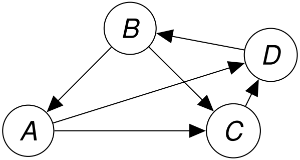

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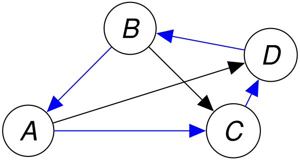

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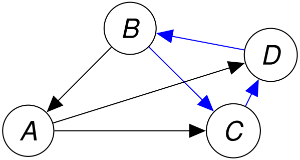

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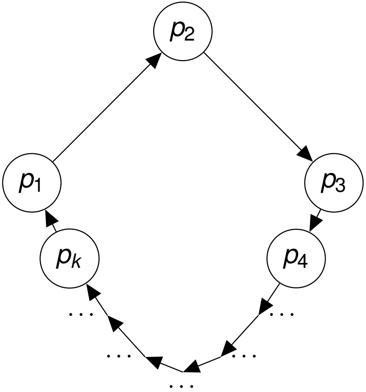

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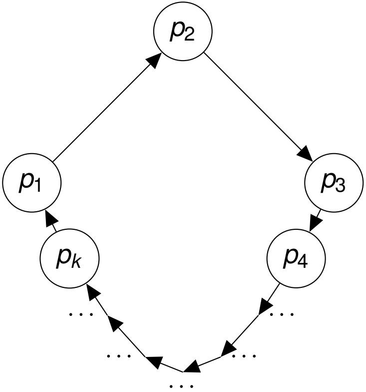

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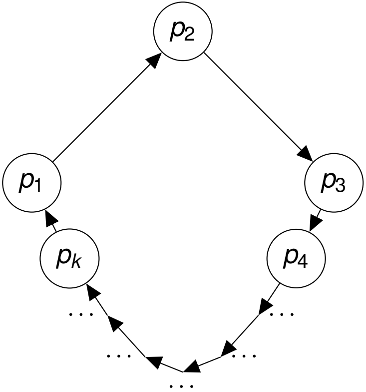

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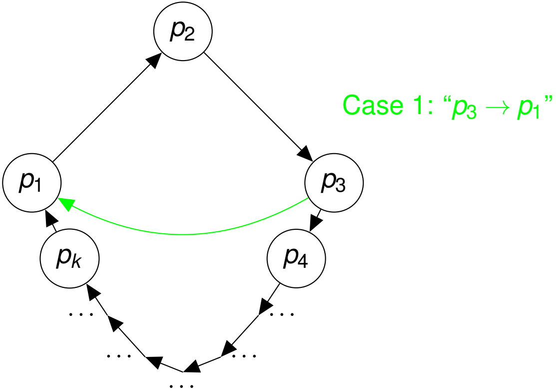

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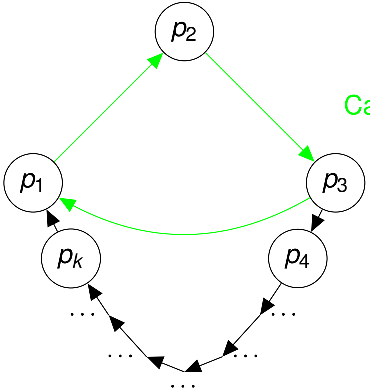

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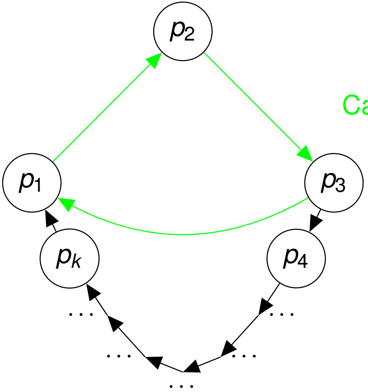

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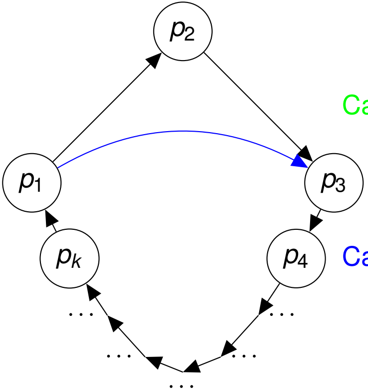

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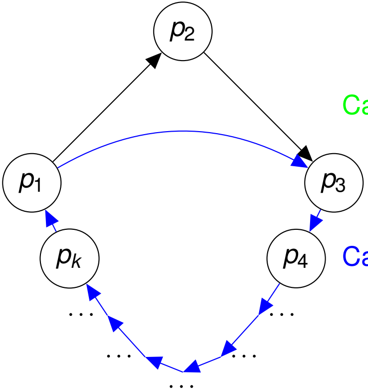

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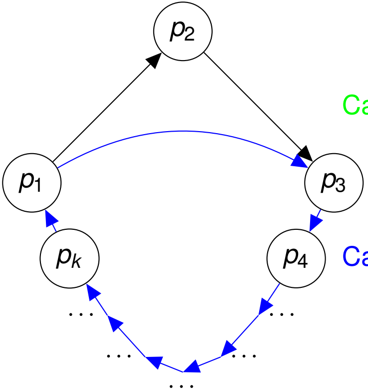


Case 1: "$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

Case 2: "$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

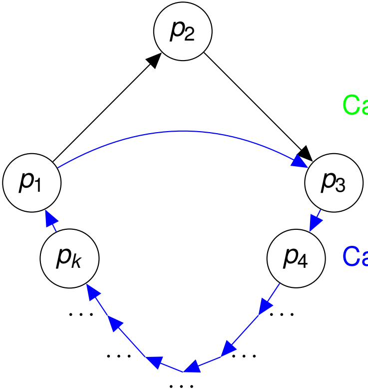
Contradicts assumption of smallest k!

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



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$\square$

Tournaments have long paths.

Def: A **round robin tournament on** n **players**: all pairs p and q play, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats q.)

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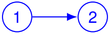

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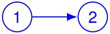

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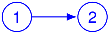

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Tournament on $n+1$ people,

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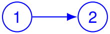

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Tournament on $n+1$ people,
Remove arbitrary person

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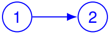

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Remove arbitrary person $\to$ yield tournament on $n-1$ people.

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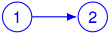

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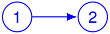

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By induction hypothesis: There is a sequence $p_1, \ldots, p_{n-1}$

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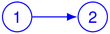

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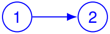

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By induction hypothesis: There is a sequence $p_1, \ldots, p_{n-1}$
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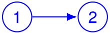

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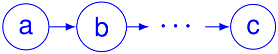

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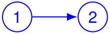

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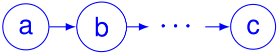

If p is big winner, put at beginning.

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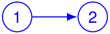

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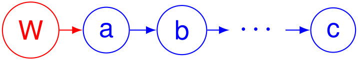

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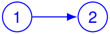

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If p is big winner, put at beginning. Big loser at end.

Tournaments have long paths.

Def: A **round robin tournament on** *n* **players**: all pairs *p* and *q* play, and either $p \to q$ (*p* beats *q*) or $q \to p$ (*q* beats *q*.)

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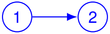

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If neither, find first place *i*, where *p* beats p_i.

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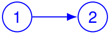

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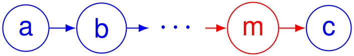

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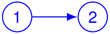

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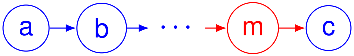

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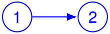

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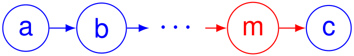

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□

Horses of the same color...

Theorem: All horses have the same color.

Horses of the same color...

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Base Case: $P(1)$ - trivially true.

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A horse in the middle in common! $1,2,3,\ldots,k,k+1$

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Second k have same color by $P(k)$.	$1, 2, 3, \ldots, k, k+1$
A horse in the middle in common!	$1, 2, 3, \ldots, k, k+1$
All k must have the same color.	$1, 2, 3, \ldots, k, k+1$

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A horse in the middle in common!

How about $P(1) \implies P(2)$?

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Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
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Of course it doesn't work.

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Of course it doesn't work.

More subtle to catch errors in proofs of correct theorems!!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

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Visitor: “I see someone has green eyes.”

Result: What happens?
(A) Nothing, no information was added.
(B) Information was added, maybe?
(C) They all leave the island.
(D) They all leave the island on day 100.

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On day 100, they all leave.

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Why?

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

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On day $n+1$, a green eyed person sees n people with green eyes.

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But they didn't leave.

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But they didn't leave.

So there must be $n + 1$ people with green eyes.

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Wait! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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On day 1, everyone knows everyone sees more than zero.

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On day 99, everyone knows no one sees 98
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Another example:

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Emperor's new clothes!

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Another example:
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  No one knows other people see that he has no clothes.

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On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
No one knows other people see that he has no clothes.
Until kid points it out.

Summary: principle of induction.

Today: More induction.

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Today: More induction.

$(P(0)$

Summary: principle of induction.

Today: More induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1))))$$

Summary: principle of induction.

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$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

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$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0

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Base Case: Prove $P(n_0)$.
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Statement is proven!

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Also Today: strengthened induction hypothesis.

Summary: principle of induction.

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Strengthen theorem statement.

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Strengthen theorem statement.
Sum of first n odds is n^2.

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Sum of first n odds is n^2.
Hole anywhere.

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Hole anywhere.
Not same as strong induction.

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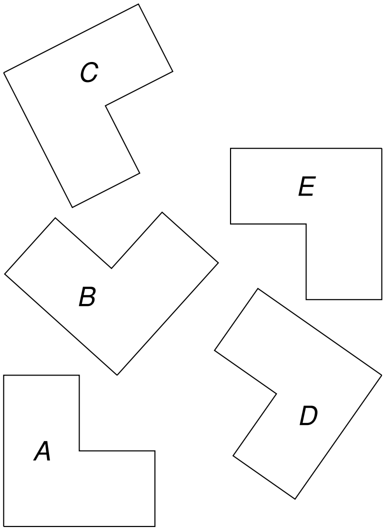
Induction $\equiv$ Recursion.

Tiling Cory Hall Courtyard.



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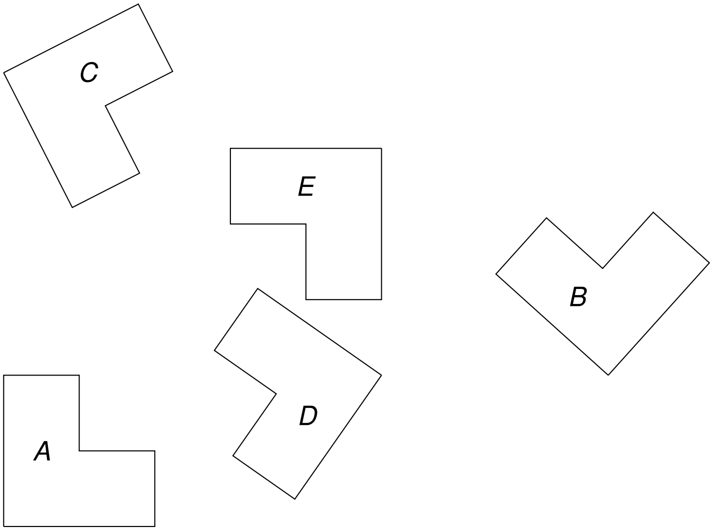

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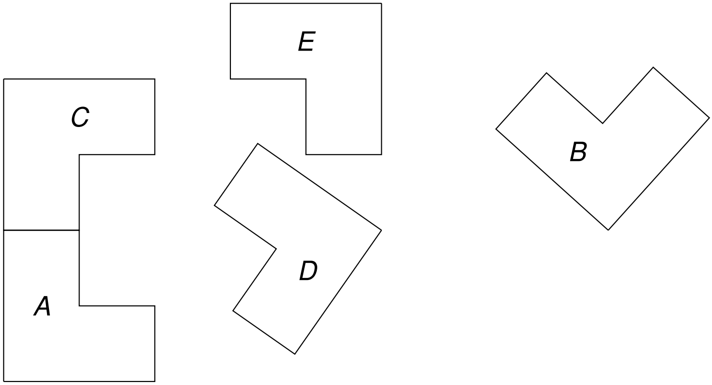

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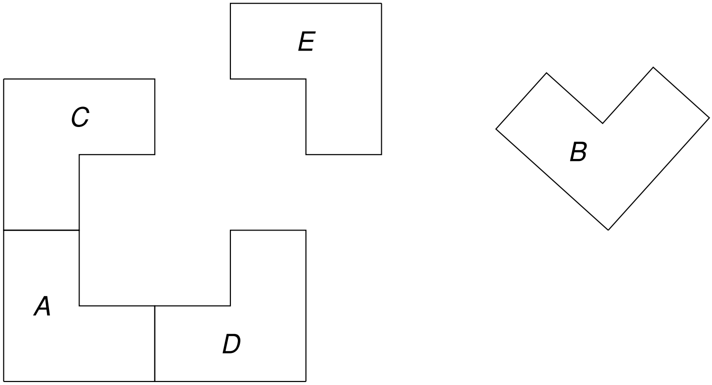

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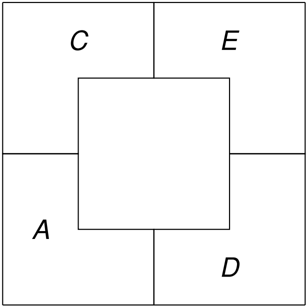

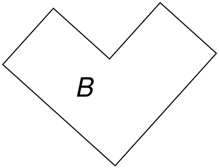

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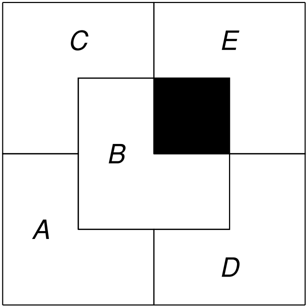

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