Today.

Quick review.

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Finish Graphs (mostly)

Degree 1 lemma.

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Proof:

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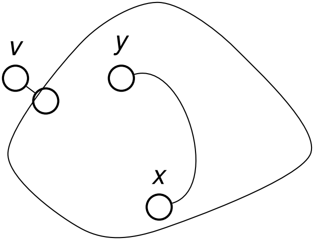

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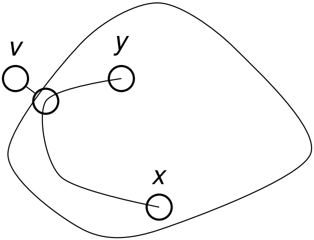
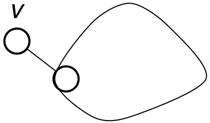


Proof of only if.

Thm:
"G connected and has $|V| - 1$ edges" $\implies$
"G is connected and has no cycles."

Proof of $\implies$:



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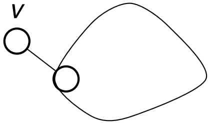

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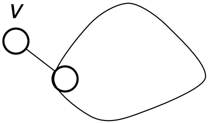

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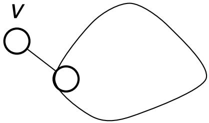

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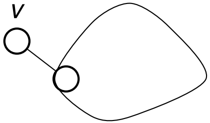

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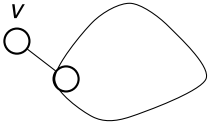

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Claim: There is a degree 1 node.

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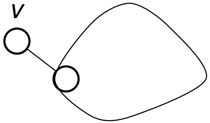

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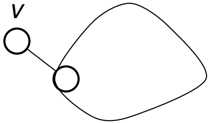

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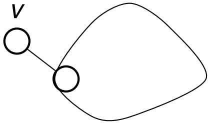

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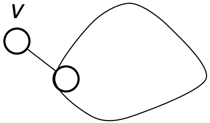

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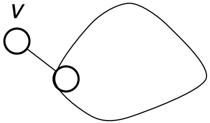

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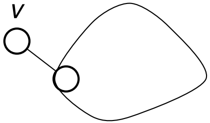

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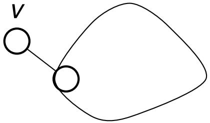

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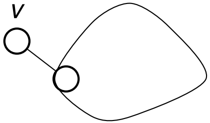

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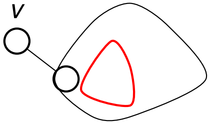

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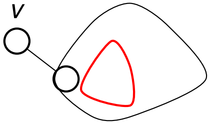

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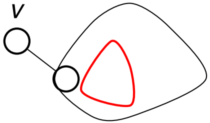

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(A) Removing a degree 1 vertex can disconnect the graph.
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(B), (C), (D) are true

Proof of "handshake" lemma.

Lemma: The sum of degrees is $2|E|$, for a graph $G = (V, E)$.
What's true?

(A) The number of edge-vertex incidences for an edge e is 2.
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Handshake lemma: sum of number of handshakes of each person is twice the number of handshakes.

Poll: Euler concepts.

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(B) Walking on unused edges, starting at v, eventually "stuck" at v.
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(D) Removing a tour leaves a connected graph.
(E) Remove set of edges E' in connected graph, connected component is incident to edge in E'
(F) A tour connecting a set of connected components, each with a Eulerian tour is really cool! This implies the graph is Eulerian.

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Only (D) is false. The rest are steps in the proof.

Lecture 6.

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Euler's Formula.

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Planar Six and then Five Color theorem.

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Planar Six and then Five Color theorem.

Types of graphs.

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Complete Graphs.
Trees (a little more.)
Hypercubes.

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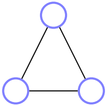

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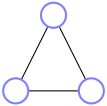

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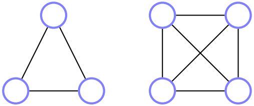

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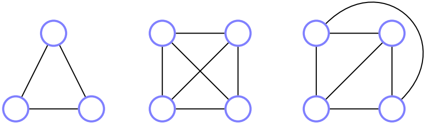

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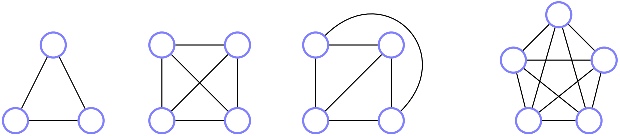

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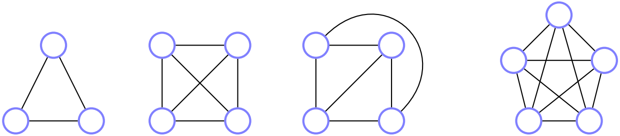

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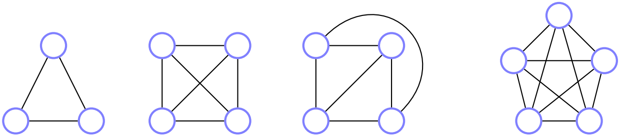

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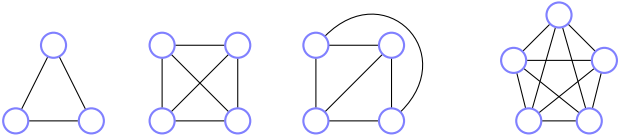

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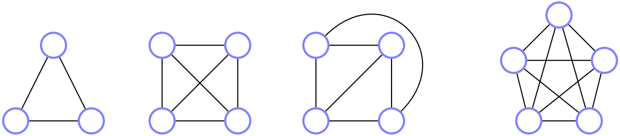

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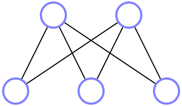

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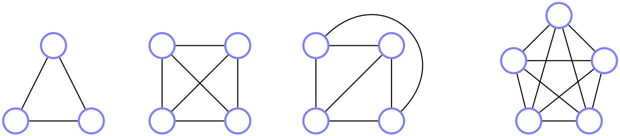

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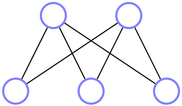

Two to three nodes, bipartite?

Planar graphs.

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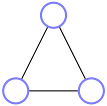 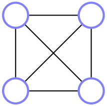 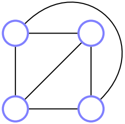 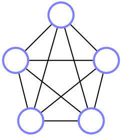

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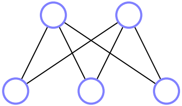 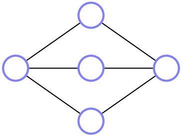

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Planar graphs.

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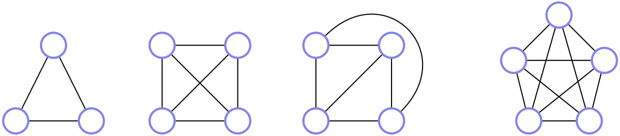

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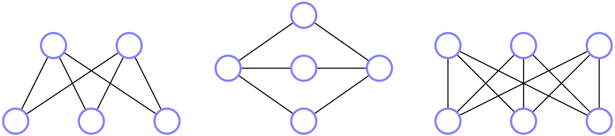

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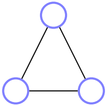
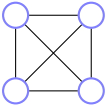
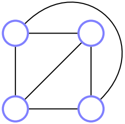
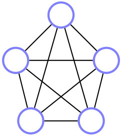

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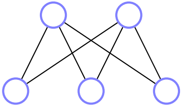
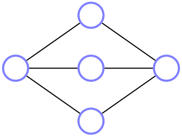
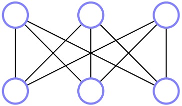

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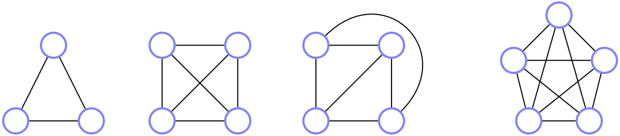

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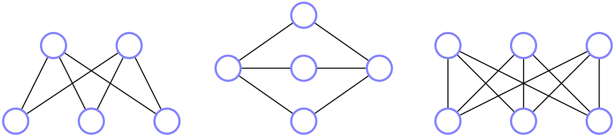

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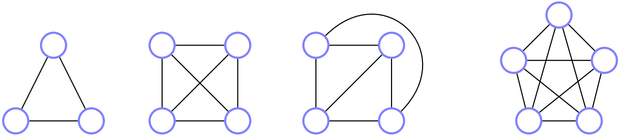

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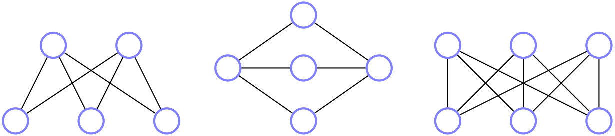

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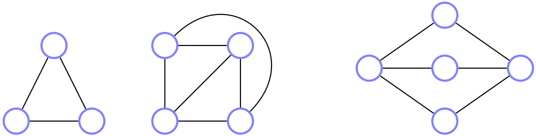

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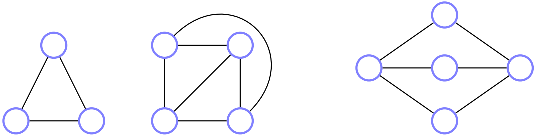

Faces: connected regions of the plane.

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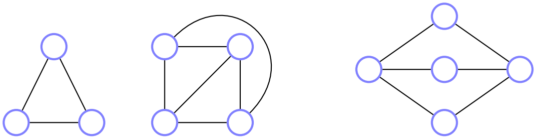

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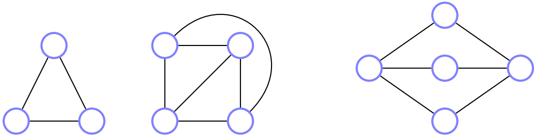

Faces: connected regions of the plane.

How many faces for

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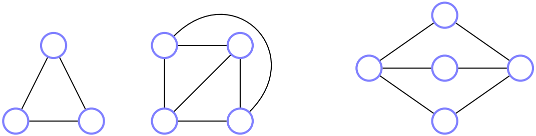

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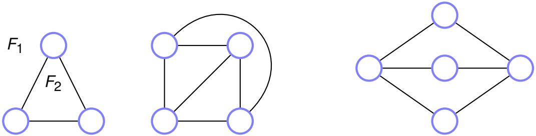

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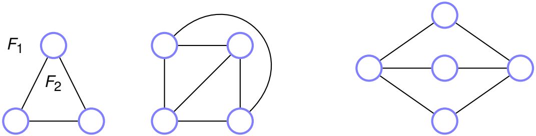

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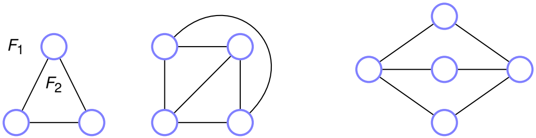

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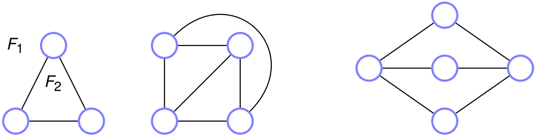

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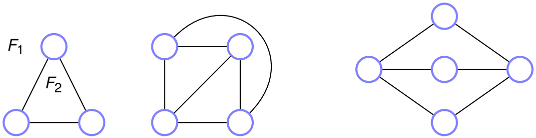

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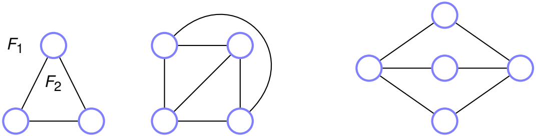

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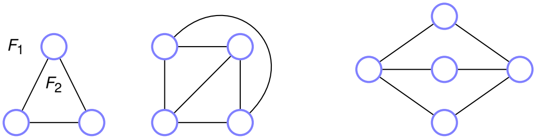

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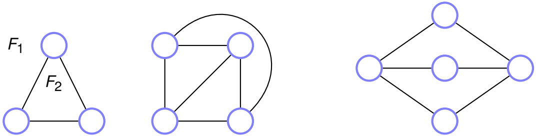

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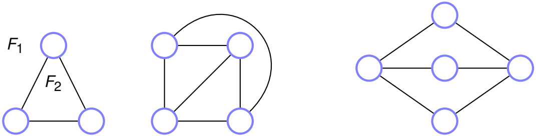

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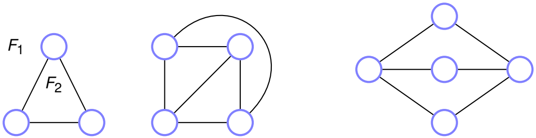

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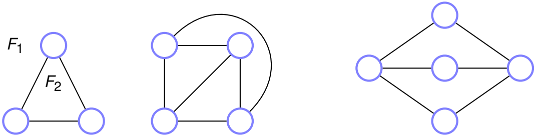

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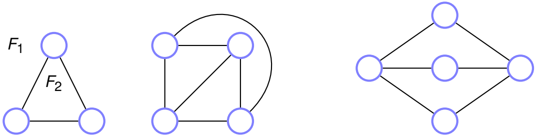

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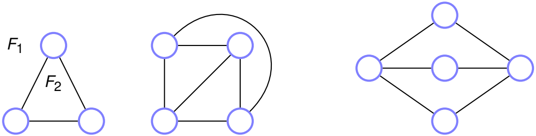

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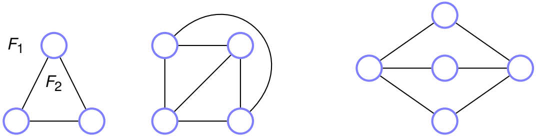

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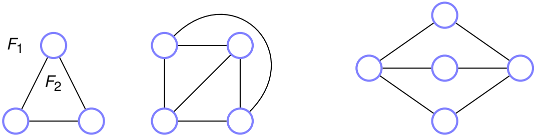

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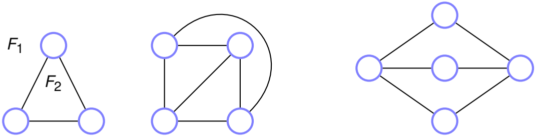

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Examples = 3!

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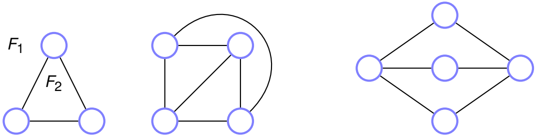

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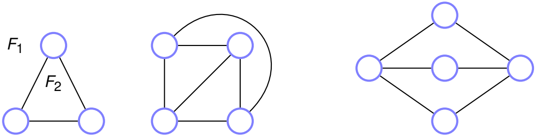

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Greeks knew formula for polyhedron.

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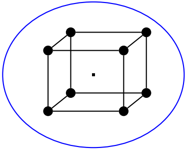

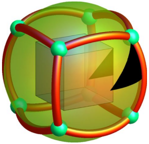

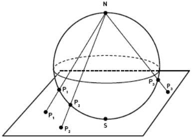

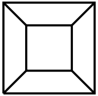

Euler and Polyhedron.

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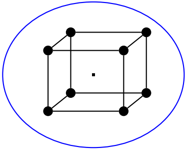

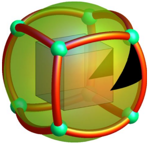

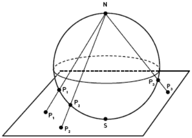

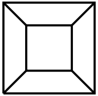

Faces?

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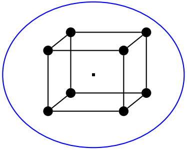 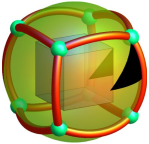 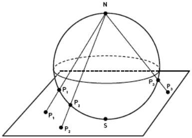

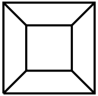

Faces? 6. Edges?

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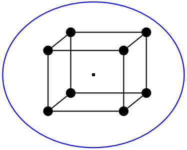

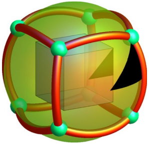

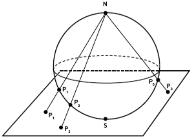

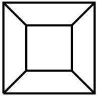

Faces? 6. Edges? 12.

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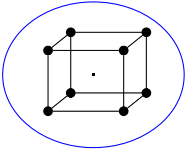
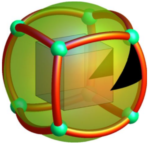
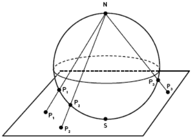

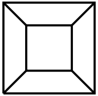

Faces? 6. Edges? 12. Vertices?

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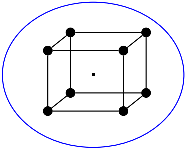
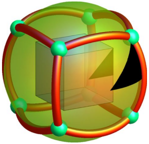
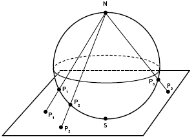
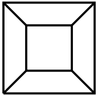




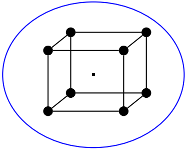
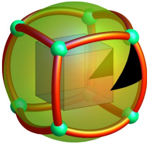
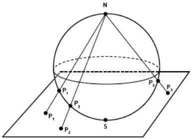
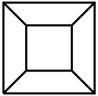




Faces? 6. Edges? 12. Vertices? 8.

Euler and Polyhedron.

Greeks knew formula for polyhedron.

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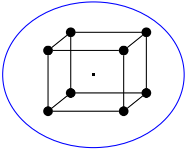
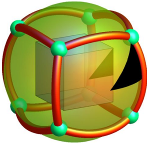
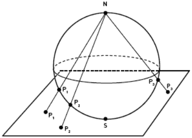

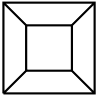

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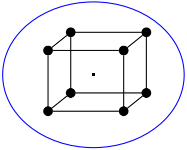
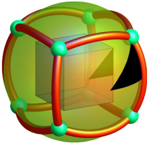
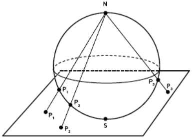

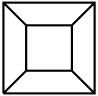

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Euler and Polyhedron.

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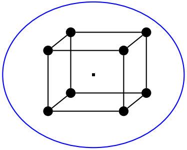

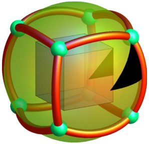

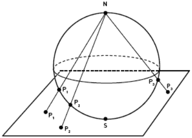

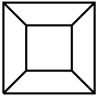

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Greeks couldn't prove it.

Euler and Polyhedron.

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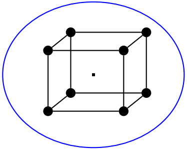
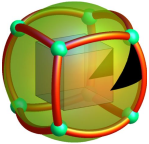
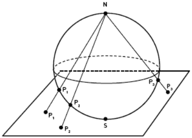
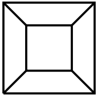

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Greeks couldn't prove it. Induction?

Euler and Polyhedron.

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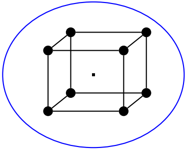

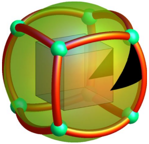

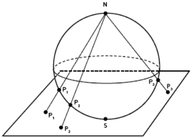

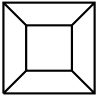

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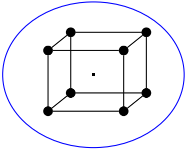
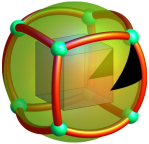
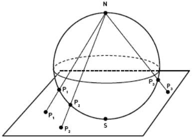
Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

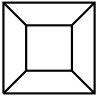
Euler and Polyhedron.

Greeks knew formula for polyhedron.









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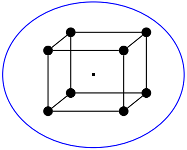
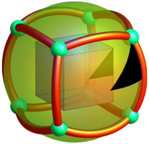
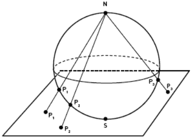
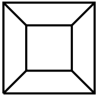
Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

Polyhedron without holes

Euler and Polyhedron.

Greeks knew formula for polyhedron.



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Polyhedron without holes $\equiv$

Euler and Polyhedron.

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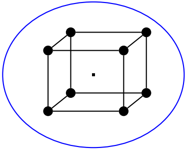
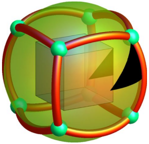
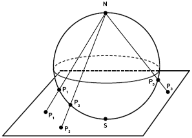

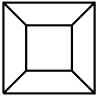

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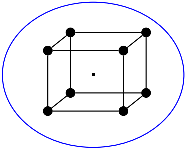
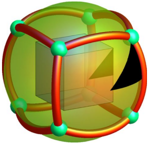
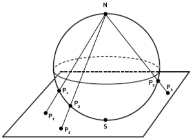
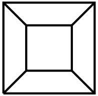
Polyhedron without holes $\equiv$ Planar graphs.

Euler and Polyhedron.

Greeks knew formula for polyhedron.



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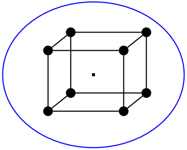
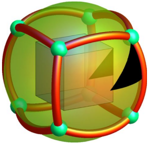
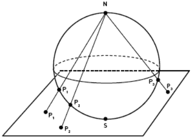
Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

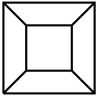
Polyhedron without holes $\equiv$ Planar graphs.

For Convex Polyhedron:

Euler and Polyhedron.

Greeks knew formula for polyhedron.



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Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

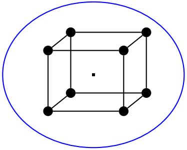
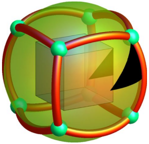
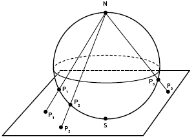
Polyhedron without holes $\equiv$ Planar graphs.

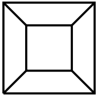
For Convex Polyhedron:

Surround by sphere.

Euler and Polyhedron.

Greeks knew formula for polyhedron.



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Polyhedron without holes $\equiv$ Planar graphs.

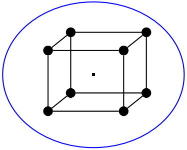
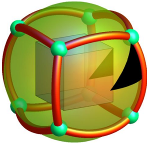
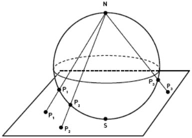
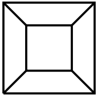
For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere:

Euler and Polyhedron.

Greeks knew formula for polyhedron.

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Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

Polyhedron without holes $\equiv$ Planar graphs.

For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere: drawing on sphere.

Euler and Polyhedron.

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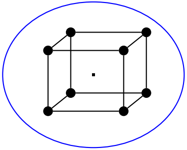

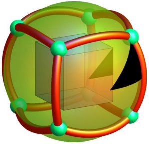

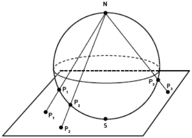

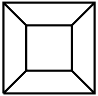

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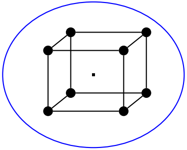
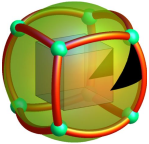
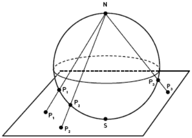

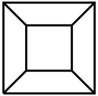

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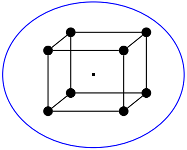
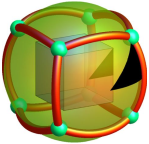
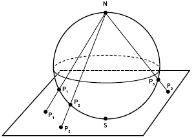

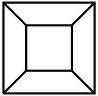

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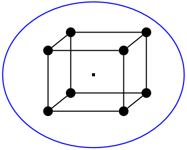
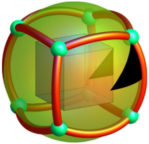
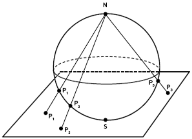
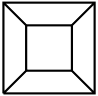

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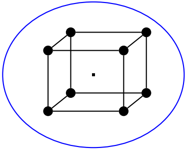 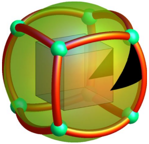 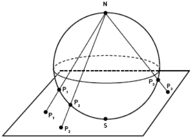 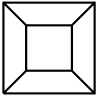

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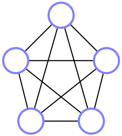
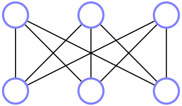
Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$





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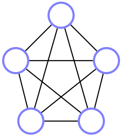

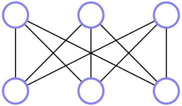

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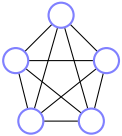

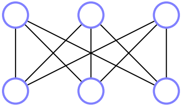

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We consider simple graphs where $v \geq 3$.

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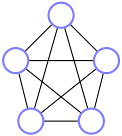

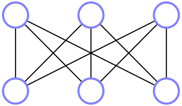

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Consider Face edge Adjacencies with multiplicities

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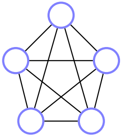

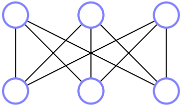

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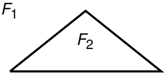

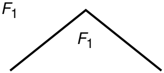

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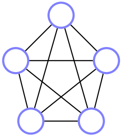

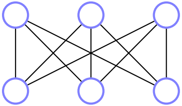

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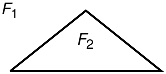

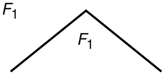

Each face is adjacent to at least three edges($v > 2$).

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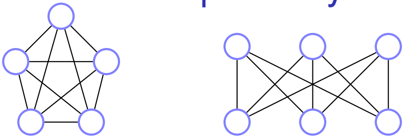

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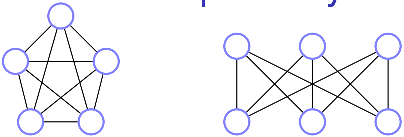

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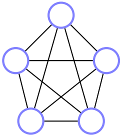

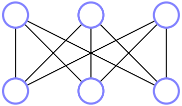

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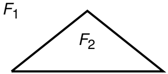

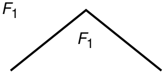

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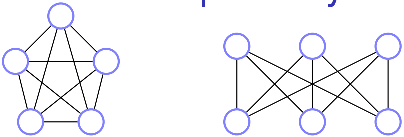

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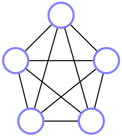

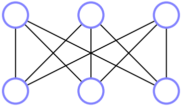

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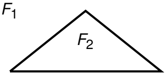

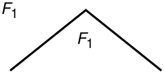

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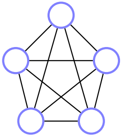

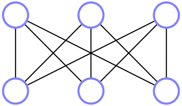

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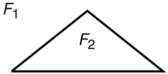

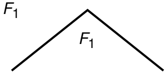

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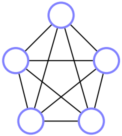

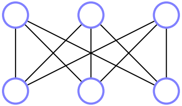

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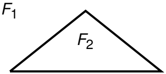

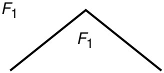

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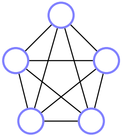

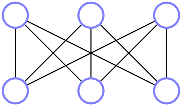

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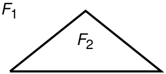
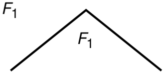
We consider simple graphs where $v \geq 3$.

Consider Face edge Adjacencies with multiplicities





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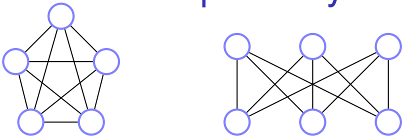

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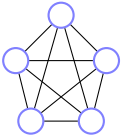

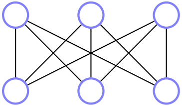

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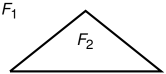

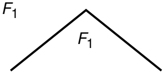

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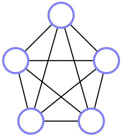

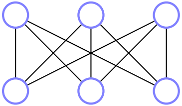

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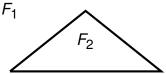

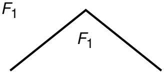

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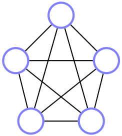

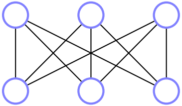

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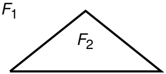

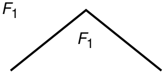

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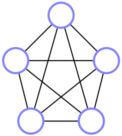

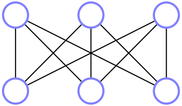

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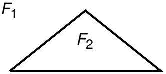

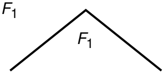

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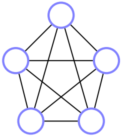
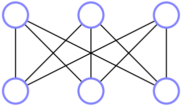
K_5

Euler and non-planarity of K_5 and $K_{3,3}$





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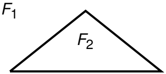

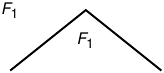

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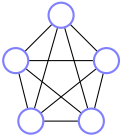

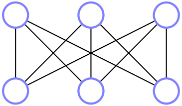

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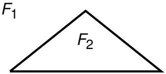

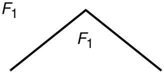

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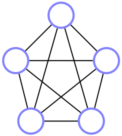

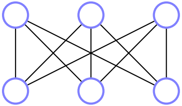

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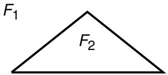

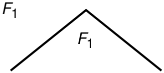

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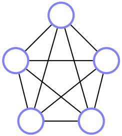

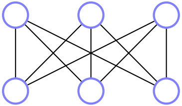

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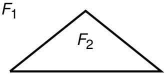

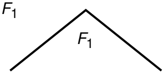

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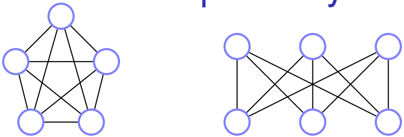

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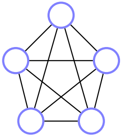

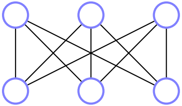

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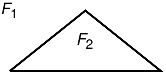

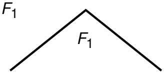

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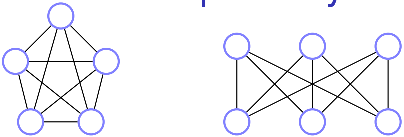

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K_5 Edges? $e = 4 + 3 + 2 + 1 = 10$. Vertices? $v = 5$.
$10 \not\leq 3(5) - 6 = 9$. $\implies K_5$ is not planar.

Planar $\implies e \leq 3v - 6$. Flow Poll.

Euler's formula: $v + f = e + 2$

Consider graph with > 2 vertices. Understand the following.

(A) Every face is incident to ≥ 3 edges.

(B) $\|$ Face-edge incidences $\| \geq 3f$

(C) Every edge is incident (with multiplicity) to 2 faces.

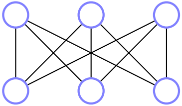
(D) $\|$Face edge incidences$\| = 2e$

(E) $3f \leq \|$Face-ege-incidences$\| = 2e$

(F) $3(e + 2 - v) <= 2e$

Conclusion: $e <= 3v - 6$

Proving non-planarity for $K_{3,3}$



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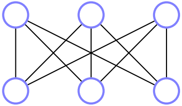

$K_{3,3}$?

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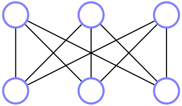

$K_{3,3}$? Edges?

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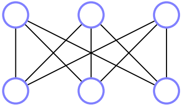

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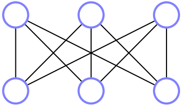

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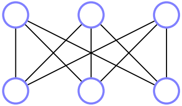

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$e \leq 3(v) - 6$ for planar graphs.

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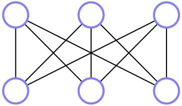

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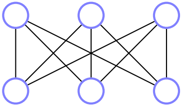
$9 \leq 3(6) - 6$?

Proving non-planarity for $K_{3,3}$



$K_{3,3}$? Edges? 9. Vertices. 6.

$e \leq 3(v) - 6$ for planar graphs.

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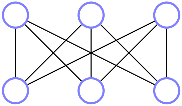

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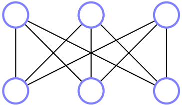

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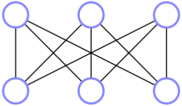
For $K_{3,3}$ every cycle is of even length or incident ≥ 4 faces.

Proving non-planarity for $K_{3,3}$



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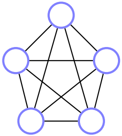
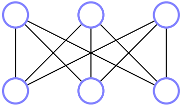
Finish in homework!

Planarity and Euler





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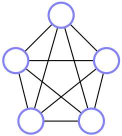
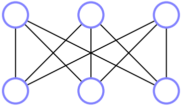
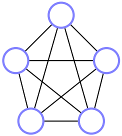
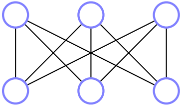




These graphs **cannot** be drawn in the plane without edge crossings.

Planarity and Euler





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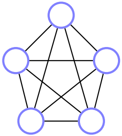
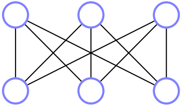
Euler's Formula: $v+f=e+2$ for any planar drawing.

Planarity and Euler

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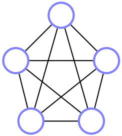
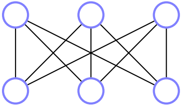

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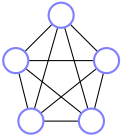

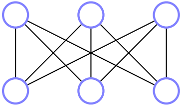

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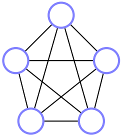

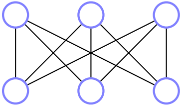

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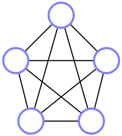

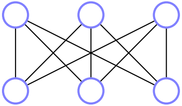

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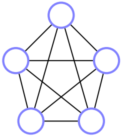

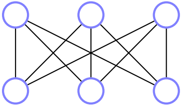

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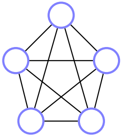

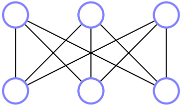

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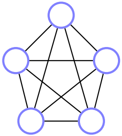

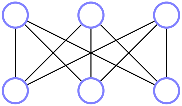

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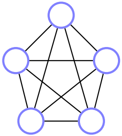

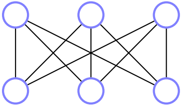

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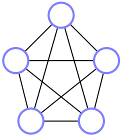
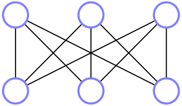
Proved absolutely no drawing can work for these graphs.

Planarity and Euler





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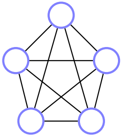
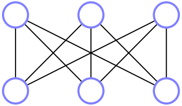
So...

Planarity and Euler





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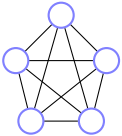

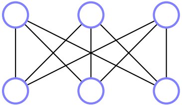

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So......so ...Cool!

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Euler's formula.

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Proof:

Euler's formula.

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Proof: Induction on e.

Euler's formula.

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Base:

Euler's formula.

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Proof: Induction on e.
Base: $e = 0$,

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Proof: Induction on e.
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Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
  If it is a tree.

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Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
If it is a tree. $e = v - 1$, $f = 1$, $v + 1 = (v - 1) + 2$. Yes.

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Proof: Induction on e.
Base: $e=0$, $v=f=1$.
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Find a cycle.

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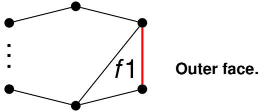

Joins two faces.

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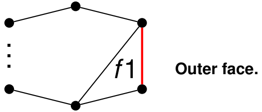

Joins two faces.
New graph: v-vertices.

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
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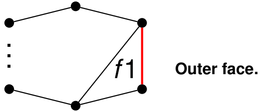

Joins two faces.
New graph: v-vertices. $e - 1$ edges.

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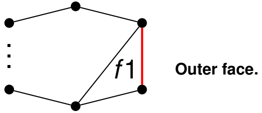

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces.

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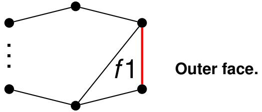

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.

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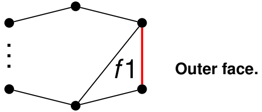

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.
$v + (f - 1) = (e - 1) + 2$ by induction hypothesis.

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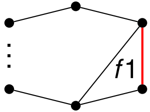

Outer face.

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.
$v + (f - 1) = (e - 1) + 2$ by induction hypothesis.
Therefore $v + f = e + 2$.

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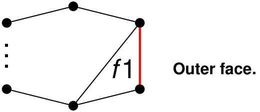

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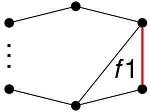

Outer face.

Joins two faces.
New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.
$v+(f-1)=(e-1)+2$ by induction hypothesis.
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Euler's formula.

Euler: Connected planar graph has $v+f=e+2$.

Proof: Induction on e.
Base: $e=0$, $v=f=1$.
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If it is a tree. $e=v-1$, $f=1$, $v+1=(v-1)+2$. Yes.
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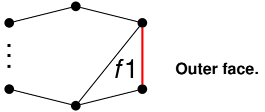

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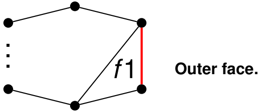

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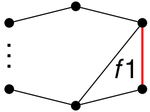

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adding edge adds face: $e \to e+1$, $f \to f+1$.

Euler's Proof.Poll.

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Steps/concepts in proof of euler's formula.

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(A) Planar drawing of tree has 1 face.
(B) Tree has $|V|-1$ edges.
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(D) face is adjacent to at least 3 edges.
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All are true and all are relevant to the proof, though (E) is more analagous than direct.

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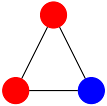

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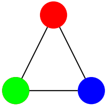

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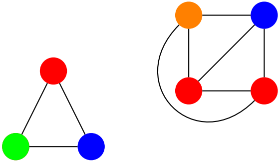

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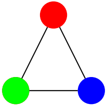

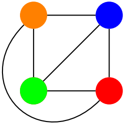

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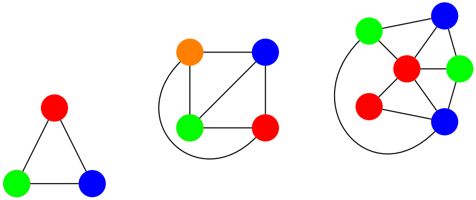

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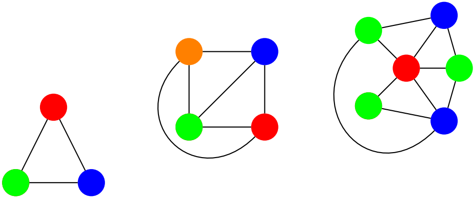

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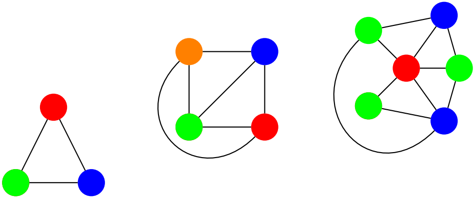

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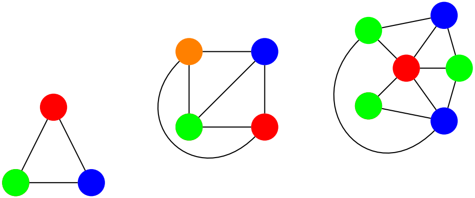

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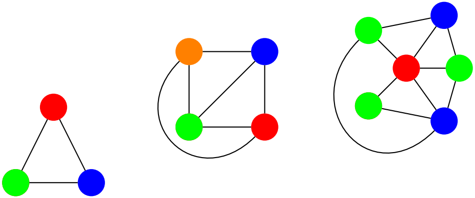

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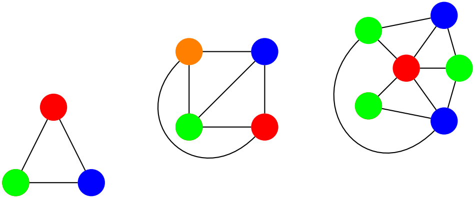

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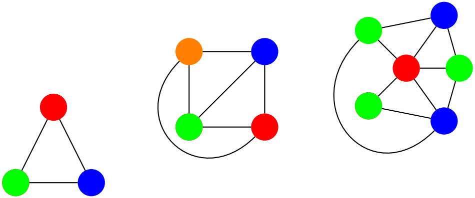

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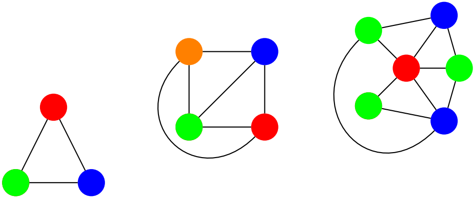

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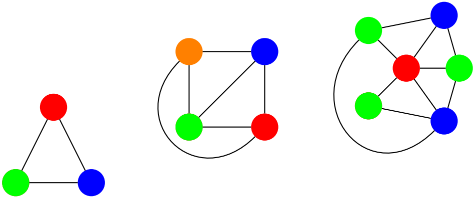

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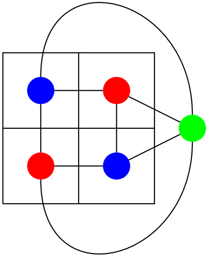
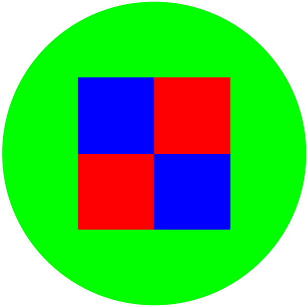
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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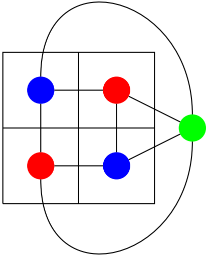

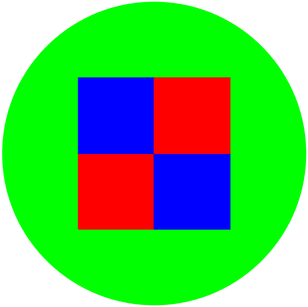

Four color theorem is about planar graphs!

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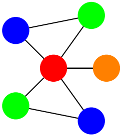
Five color theorem: prelimnary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



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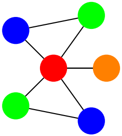

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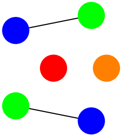

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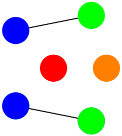

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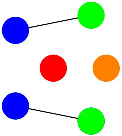

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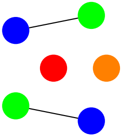

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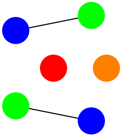

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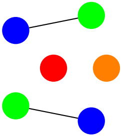

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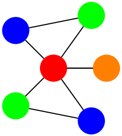

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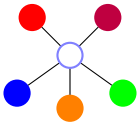

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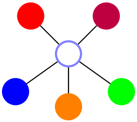

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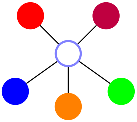

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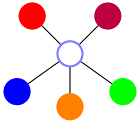

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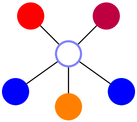

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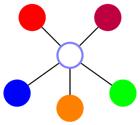
Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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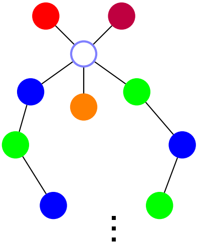

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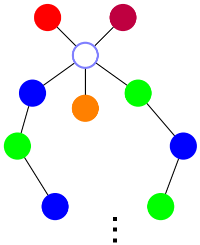

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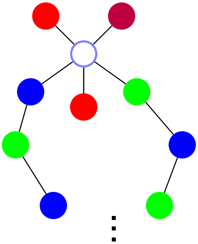

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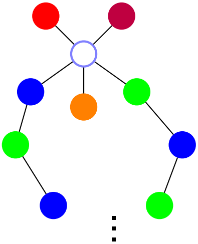

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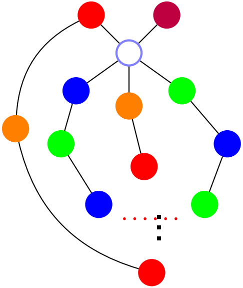

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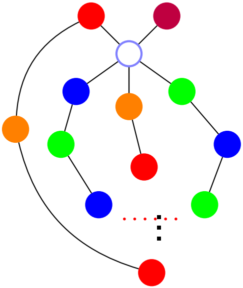

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Planar.

Five color theorem

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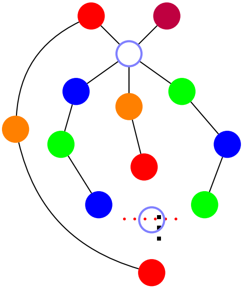
Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar. $\Longrightarrow$ paths intersect at a vertex!

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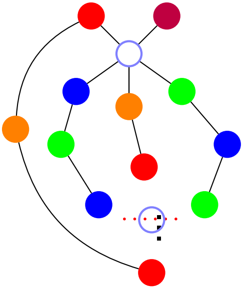
Proof: Again with the degree 5 vertex. Again recurse.



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What color is it?

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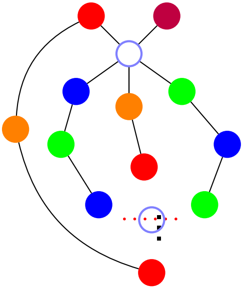

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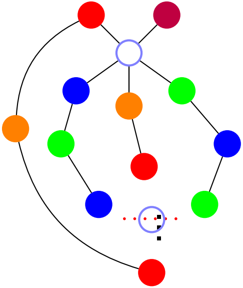

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.

Five color theorem

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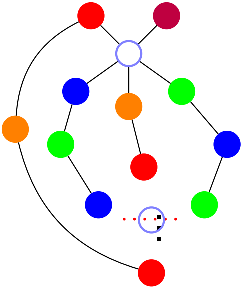

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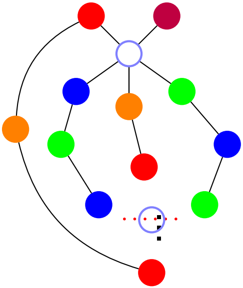

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction.

Five color theorem

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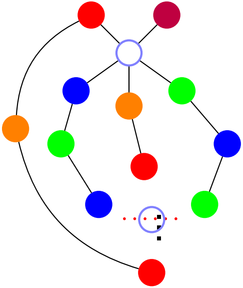
Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

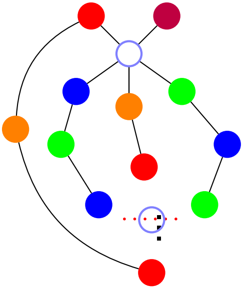
Contradiction. Can recolor one of the neighbors.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Done. Unless blue-green path to blue.
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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

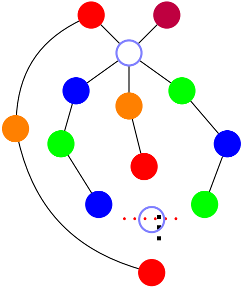
Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex!

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex! □

5 color theorem. Flow poll.

Steps/ideas in 5-color theorem.
(A) There is a degree 5 vertex cuz Euler.
(B) Take subgraph of first and third colors, recolor first components.
(C) If a third's component is different, switched coloring is good.
(D) Subgraph of second and fourth colors, can recolor, recolor second component.
(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

5 color theorem. Flow poll.

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(A) There is a degree 5 vertex cuz Euler.
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(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

All steps in proof!

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

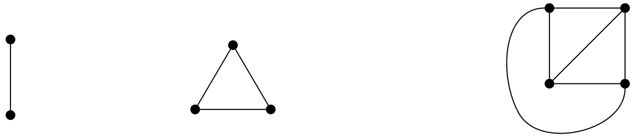
Proof: Not Today!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

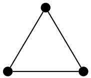
Proof: Not Today!

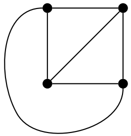
Complete Graph.



K_n complete graph on n vertices.

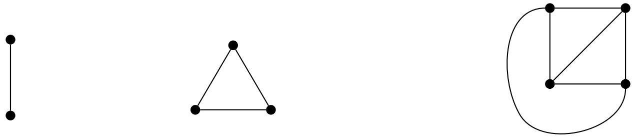
Complete Graph.





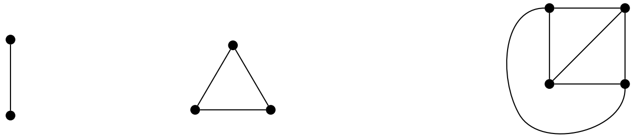
K_n complete graph on n vertices.
All edges are present.

Complete Graph.



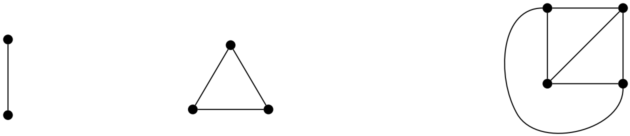
K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.

Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

Complete Graph.



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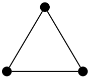

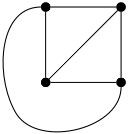

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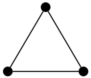
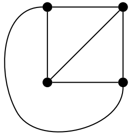
How many edges?

Complete Graph.





K_n complete graph on n vertices.

All edges are present.

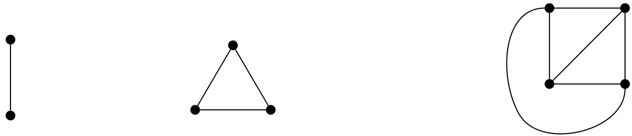
Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Complete Graph.



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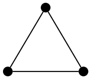
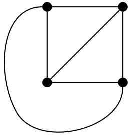
Sum of degrees is $n(n-1)$

Complete Graph.

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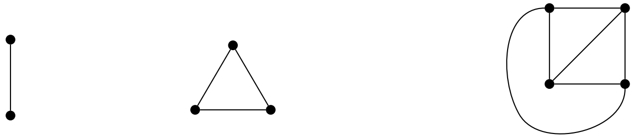
How many edges?

Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1) = 2|E|$

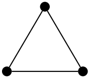
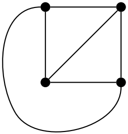
Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1) = 2|E|$
$\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.

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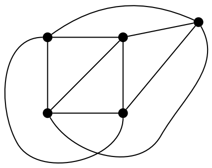
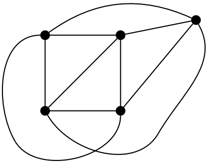
K_4 and K_5



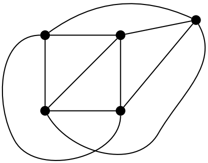
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

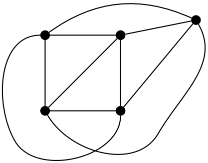


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! We did!

Hypercubes.

Complete graphs, really connected!

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees,

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

but just falls apart!

Hypercubes.

Complete graphs, really connected! But lots of edges.

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Hypercubes.

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Trees, few edges. $(|V|-1)$

but just falls apart!

Hypercubes. Really connected. $|V|\log|V|$ edges!

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

but just falls apart!

Hypercubes. Really connected. $|V|\log|V|$ edges!

Also represents bit-strings nicely.

Hypercubes.

Complete graphs, really connected! But lots of edges.

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$G = (V,E)$

Hypercubes.

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Trees, few edges. $(|V|-1)$

but just falls apart!

Hypercubes. Really connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

$G=(V,E)$
$|V|=\{0,1\}^n$,

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

but just falls apart!

Hypercubes. Really connected. $|V|\log|V|$ edges!

Also represents bit-strings nicely.

$G=(V,E)$

$|V|=\{0,1\}^n$,

$|E|=\{(x,y)|x$ and y differ in one bit position.$\}$

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

but just falls apart!

Hypercubes. Really connected. $|V|\log|V|$ edges!

Also represents bit-strings nicely.

$G=(V,E)$

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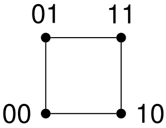

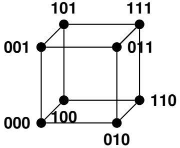

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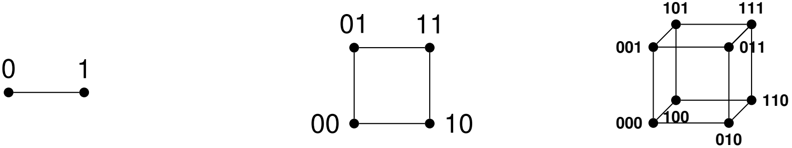

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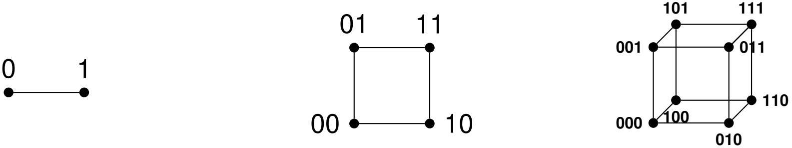

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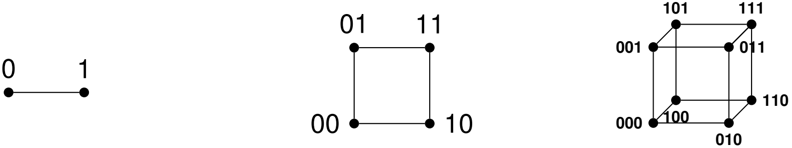

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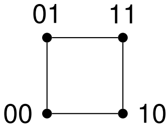

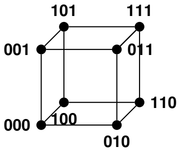

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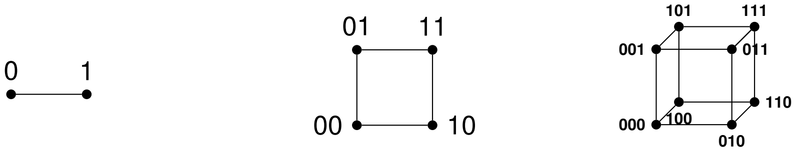

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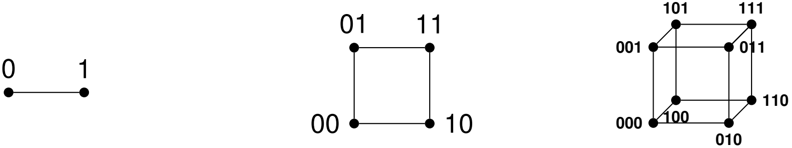

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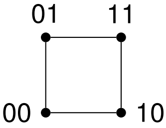

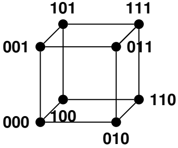

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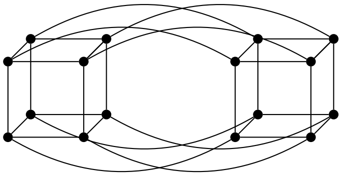

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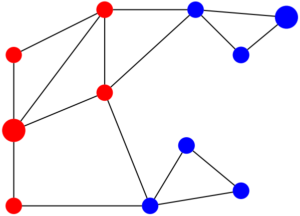
Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Cuts in graphs.



S is red, $V - S$ is blue.

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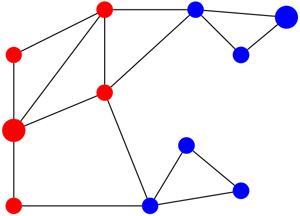

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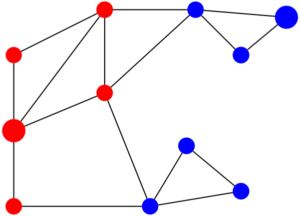

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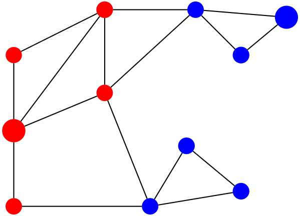

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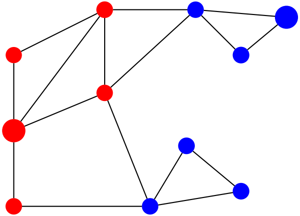

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Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

Proof of Large Cuts.

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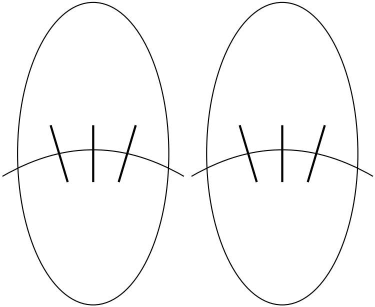

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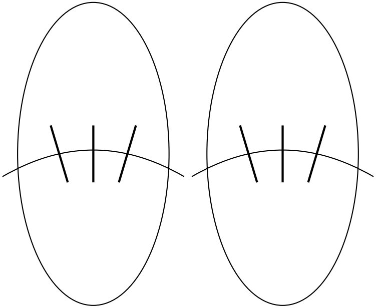

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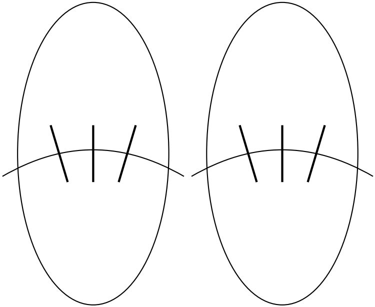

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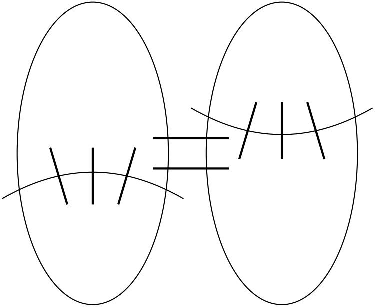

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Proof: Induction Step. Case 2.

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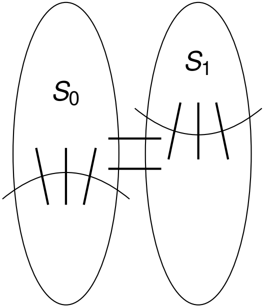

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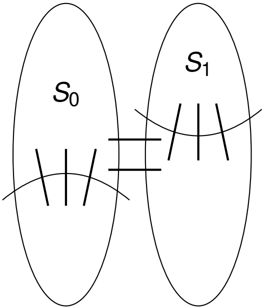

$|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

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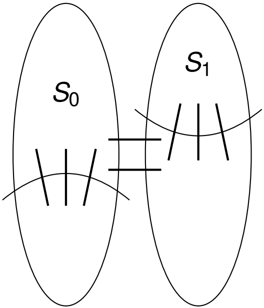

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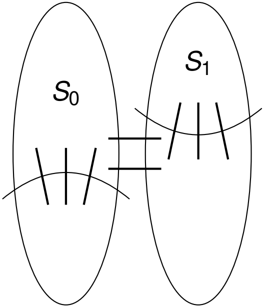

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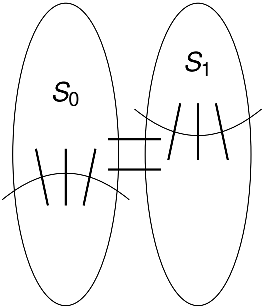

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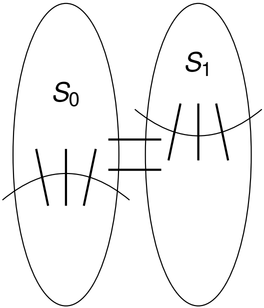

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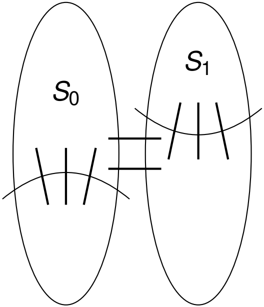

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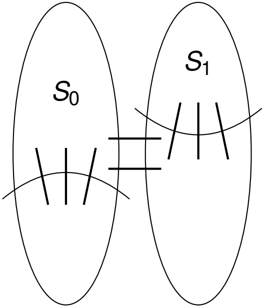

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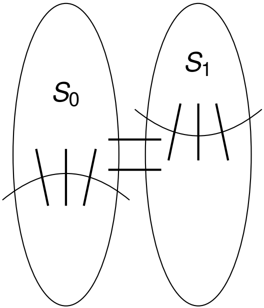

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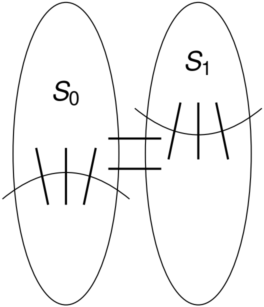

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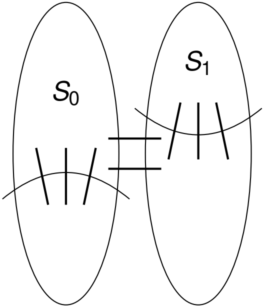

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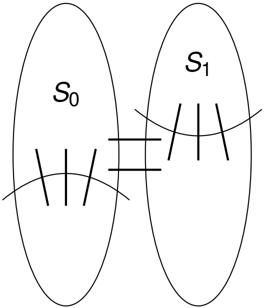

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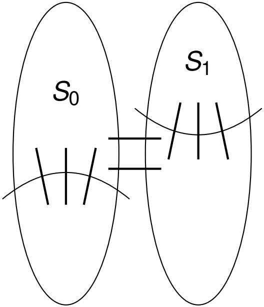

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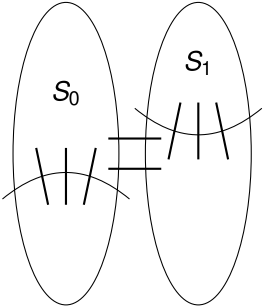

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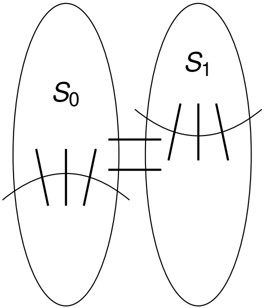

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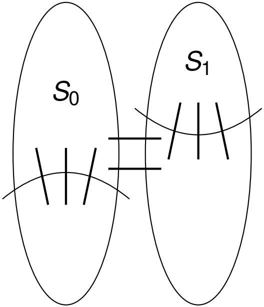

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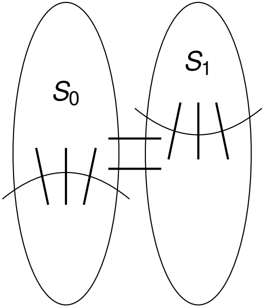

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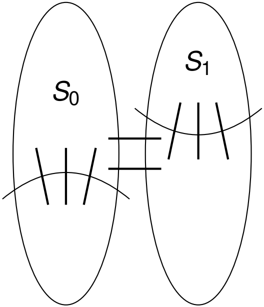

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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Graphs:
Trees: sparsest connected.
Complete:densest
Hypercube: middle.