Lecture 7. Outline.

1. Modular Arithmetic.

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   Clock Math!!!

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2. Inverses for Modular Arithmetic: Greatest Common Divisor.

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   Division!!!

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1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
3. Euclid's GCD Algorithm.

Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
3. Euclid's GCD Algorithm.
   A little tricky here!

Modular Arithmetic.

Applications: cryptography, error correction.

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

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Proof:

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$x = ad$, $y = bd$,

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Proof:
$x = ad$, $y = bd$,
$(x-y) = (ad-bd) = d(a-b) \implies d|(x-y)$.

$\square$

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

Proof:
$x = ad$, $y = bd$,
$(x-y) = (ad-bd) = d(a-b) \implies d|(x-y)$.

$\square$

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

Proof:
$x = ad$, $y = bd$,
$(x-y) = (ad-bd) = d(a-b) \implies d|(x-y)$.
□

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. □
(Uniqueness? Later.)

Poll

What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler's formula.
(C) The definition of a prime number.
(D) Euclid's Lemma.

Poll

What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler's formula.
(C) The definition of a prime number.
(D) Euclid's Lemma.

(A) and (C)

Next Up.

Modular Arithmetic.

Clock Math

If it is 1:00 now.

Clock Math

If it is 1:00 now.
What time is it in 2 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
  Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.
- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
- Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
- $101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.
- Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
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Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

Clock Math

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What time is it in 2 hours? 3:00!
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Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$
(Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

This is Thursday is September 15, 2022.

Day of the week.

This is Thursday is September 15, 2022.

What day is it a year from now?

Day of the week.

This is Thursday is September 15, 2022.
What day is it a year from now? on September 15, 2022?

Day of the week.

This is Thursday is September 15, 2022.
  What day is it a year from now? on September 15, 2022?
    Number days.

Day of the week.

This is Thursday is September 15, 2022.

What day is it a year from now? on September 15, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

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What day is it a year from now? on September 15, 2022?
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Day of the week.

This is Thursday is September 15, 2022.

- What day is it a year from now? on September 15, 2022?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

Day of the week.

This is Thursday is September 15, 2022.

What day is it a year from now? on September 15, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then.

Day of the week.

This is Thursday is September 15, 2022.
- What day is it a year from now? on September 15, 2022?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
- 5 days from then. day 9

Day of the week.

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What day is it a year from now? on September 15, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2

Day of the week.

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Number days.

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Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

Day of the week.

This is Thursday is September 15, 2022.

What day is it a year from now? on September 15, 2022?

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Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then.

Day of the week.

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5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29

Day of the week.

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5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1.

Day of the week.

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5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

Day of the week.

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two days are equivalent up to addition/subtraction of multiple of 7.

Day of the week.

This is Thursday is September 15, 2022.

- What day is it a year from now? on September 15, 2022?
  - Number days.
    - 0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

- 5 days from then. day 9 or day 2 or Tuesday.
- 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then

Day of the week.

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two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1

Day of the week.

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11 days from then is day 1 which is Monday!

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11 days from then is day 1 which is Monday!

What day is it a year from then?

Day of the week.

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25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year.

Day of the week.

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What day is it a year from now? on September 15, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.

Day of the week.

This is Thursday is September 15, 2022.

What day is it a year from now? on September 15, 2022?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Day of the week.

This is Thursday is September 15, 2022.
What day is it a year from now? on September 15, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:

Day of the week.

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Number days.
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25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.

Day of the week.

This is Thursday is September 15, 2022.
  What day is it a year from now? on September 15, 2022?
    Number days.
      0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. $29 = (7)4 + 1$
    two days are equivalent up to addition/subtraction of multiple of 7.
    11 days from then is day 1 which is Monday!

What day is it a year from then?
  Next year is not a leap year. So 365 days from then.
  Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.

Day of the week.

This is Thursday is September 15, 2022.
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Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7

Day of the week.

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two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5.

Day of the week.

This is Thursday is September 15, 2022.

What day is it a year from now? on September 15, 2022?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

Day of the week.

This is Thursday is September 15, 2022.
  What day is it a year from now? on September 15, 2022?
    Number days.
      0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. $29 = (7)4 + 1$
    two days are equivalent up to addition/subtraction of multiple of 7.
    11 days from then is day 1 which is Monday!

What day is it a year from then?
  Next year is not a leap year. So 365 days from then.
  Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
  369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$
    or September 15, 2022 is a Friday.

Day of the week.

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What day is it a year from now? on September 15, 2022?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
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11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$
or September 15, 2022 is a Friday.

Years and years...

80 years?

Years and years...

80 years? 20 leap years.

Years and years...

80 years? 20 leap years. 366×20 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

Years and years...

80 years?   20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60$

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

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Further Simplify Calculation:

Years and years...

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Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.

Years and years...

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20 has remainder 6 when divided by 7.
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Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.

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Or Day 6.

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Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.

Or Day 6. September 15, 2102 is Saturday.

“Reduce” at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.

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...or x and y have the same remainder w.r.t. m.

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Mod 7 equivalence or *residue* classes:

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Mod 7 equivalence or *residue* classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$

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Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

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or " $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$

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If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer j.
Therefore,

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Therefore, $a+b = c+d+(k+j)m$

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Therefore, $a+b = c+d+(k+j)m$ and since $k+j$ is integer.

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Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
$\implies a + b \equiv c + d \pmod{m}$.

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Therefore, $a+b = c+d+(k+j)m$ and since $k+j$ is integer.
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Can calculate with representative in $\{0,\ldots,m-1\}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

Notation

$x \pmod m$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12$

Notation

$x \pmod{m}$ or $\mod(x,m)$
- remainder of x divided by m in $\{0,\ldots,m-1\}$.

$\mod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = 4$

Notation

$x \pmod{m}$ or $\bmod(x,m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\bmod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\bmod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \not{4} = 5$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.

$a \equiv b \pmod{m}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$

- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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Modulus is m

Notation

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- remainder of x divided by m in $\{0,\ldots,m-1\}$.

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Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv$

Notation

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$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3$

Notation

$x \pmod m$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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Work in this system.
$a \equiv b \pmod m$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10$

Notation

$x \pmod{m}$ or $\mod(x, m)$
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$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

Notation

$x \pmod{m}$ or $\mod(x,m)$
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Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3+3 \equiv 3+10 \pmod{7}$.

$6 =$

Notation

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3$

Notation

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10$

Notation

$x \pmod{m}$ or $\bmod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\bmod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Notation

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$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

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Generally, not $6 \pmod{7} = 13 \pmod{7}$.

Notation

$x \pmod m$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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Check! $4(3) = 12 = 5 \pmod{7}$.

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$8k \not\equiv 1 \pmod{12}$ for any k.

Poll

Mark true statements.
(A) Mutliplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of $((n-1) \pmod n) = ((n-1) \pmod n)$).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of $4 = -1 \pmod 5$.
(E) $(-1)x(-1) = 1$. Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.

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(C) is false. 0.5 has no meaning in arithmetic modulo 5.

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Thm:
If greatest common divisor of x and m, $\gcd(x, m)$, is 1, then x has a multiplicative inverse modulo m.

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$S=\{0(4),1(4),2(4),3(4),4(4),5(4)\}=\{0,4,8,12,16,20\}$

reducing $\pmod 6$

$S=\{0,4,2,0,4,2\}$

Proof review. Consequence.

Thm: If $\gcd(x, m) = 1$, then x has a multiplicative inverse modulo m.

Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

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$5x=3 \pmod 6$ What is x? Multiply both sides by 5.
$x=15$

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Very different for elements with inverses.

Proof Review 2: Bijections.

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Therefore $d| \mod(x,y)$.

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Before discussing running time of gcd procedure...

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one million or 1,000,000!

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Poll.

Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

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Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

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Assume $\log_2 1{,}000{,}000$ **is 20 to the nearest integer.**
Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).

Poll

Which are correct?

(A) gcd(700,568) = gcd (568,132)
(B) gcd(8,3) = gcd(3,2)
(C) gcd(8,3) = 1
(D) gcd(4,0) = 4

Poll

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euclid(700,568)
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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

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euclid(700,568)
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Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

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At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Runtime Proof.

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(define (euclid x y)
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Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

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□

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Didn’t necessarily need to do gcd.

Runtime proof still works.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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For x and m, if $\gcd(x,m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer k.

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Division? Multiply by multiplicative inverse.

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Know if there is an inverse, but how do we find it?

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Euclid's Extended GCD Theorem:
For any x, y there are integers a, b where

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What is multiplicative inverse of x modulo m?

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By extended GCD theorem, when $\gcd(x, m) = 1$.

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By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

Extended GCD

Euclid's Extended GCD Theorem:
For any x, y there are integers a, b where

$$ax + by = d \qquad \text{where } d = gcd(x, y).$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

Extended GCD

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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$a = 3$ and $b = -1$.

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$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of x and y..?

```
gcd(35,12)
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Get 1 from 12 and 11.
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Get 11 from 35 and 12 and plugin.... Simplify.

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  gcd(12, 11)  ;;  gcd(12, 35%12)
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.

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Example:

```
ext-gcd(35,12)
```

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Example:

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ext-gcd(35,12)
  ext-gcd(12, 11)
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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ext-gcd(35,12)
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
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         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
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ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = \; 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
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    ext-gcd(35,12)
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      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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```

Theorem: Returns (d,a,b), where $d = gcd(a,b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x, 0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod (x, y))$ returns (d, a, b) with
$$d = ay + b(\mod (x, y))$$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
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ext-gcd(x, y) calls **ext-gcd**$(y, \mod(x, y))$ so

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$$d = ay + b \cdot (\mod(x, y))$$

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Base: ext-gcd($x, 0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

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$$\begin{aligned} d &= ay + b \cdot (\mod(x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \end{aligned}$$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x, 0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
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ext-gcd(x, y) calls **ext-gcd**($y, \mod (x, y)$) so

$$\begin{aligned} d &= ay + b \cdot (\mod (x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b) y \end{aligned}$$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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Ind hyp: **ext-gcd**$(y, \mod(x, y))$ returns (d, a, b) with
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ext-gcd(x, y) calls **ext-gcd**$(y, \mod(x, y))$ so

$$\begin{aligned} d &= ay + b \cdot (\mod(x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds! □

[1] Assume d is $gcd(x, y)$ by previous proof.

Review Proof: step.

Prove: returns (d, A, B) where $d = Ax + By$.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Review Proof: step.

Prove: returns (d, A, B) where $d = Ax + By$.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

Review Proof: step.

Prove: returns (d, A, B) where $d = Ax + By$.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Review Proof: step.

Prove: returns (d, A, B) where $d = Ax + By$.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.

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Confirm:

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Confirm: $-119+120=1$

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Very different from elementary school: try 1, try 2, try 3...

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?

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≤ 80 divisions.

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Internet Security: Next Week.